Automatica 68 (2016) 266–276

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On asymptotic stability of linear time-varying systems☆

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ARTICLE INFO

Article history:
Received 12 June 2014
Received in revised form
14 October 2015
Accepted 8 December 2015
Available online 27 February 2016

Keywords:
Linear time-varying systems
Asymptotic stability
Exponential stability
Stable functions
Differential Lyapunov inequalities

ABSTRACT

This paper is concerned with asymptotic stability analysis of linear time-varying (LTV) systems. With the help of the notion of stable functions, some differential Lyapunov inequalities (DLIs) based necessary and sufficient conditions are derived for testing asymptotic stability, exponential stability and uniformly exponential stability of general LTV systems. With the help of the concept of (non-uniformly) exponential stability, a class of upper-triangular LTV systems is carefully investigated based on the developed stability analysis approaches. A couple of numerical examples with some of them borrowed from the literature is provided to illustrate the effectiveness of the proposed theoretical results.

1. Introduction

The stability of dynamics systems is the most important criterion in system design (Harris & Miles, 1980). Hence stability of linear time-varying (LTV) systems including linear time-invariant ones as special cases has received considerable attention during the past several decades (see Anderson & Moore, 1969, Haken & Naylor, 1966, Malek-zavarei, 1978, Mazenc, Malisoff, & Niculescu, 2014, Mullhaupt, Buccieri, & Bonvin, 2007, Sun, 2007, Zhou, Cai, & Duan, 2013 and the references therein). There has been considerable development of the state space approach to stability theory of linear time-invariant systems in the past several decades, while the corresponding status of LTV systems is comparatively retarded (Harris & Miles, 1980). This is because, as pointed out in Harris and Miles (1980), state transition matrices of LTV systems, which are impossible to be derived except for very particular cases (see, for example, Wu, Horowitz, & Dennison, 1975), are generally needed to ascertain the properties of stability, while for linear time-invariant systems such properties can be determined directly (or indirectly) in terms of the system parameters. Particularly, the stability of a linear time-invariant system is totally determined by the locations of the eigenvalues of the system matrix, while the stability of an LTV system cannot be linked with the locations of the eigenvalues of its system matrix (Rugh, 1996; Wu, 1974). Consequently, it seems that simple yet necessary and sufficient conditions guaranteeing stability of general LTV systems are not available, except for the very special case that the system is periodic (Zhou & Duan, 2012; Zhou, Hou, & Duan, 2013).

On the other hand, in contrast to linear time-invariant systems which only have two kinds of stability concepts, namely, (Lyapunov) stability and asymptotic stability, there are several different stability concepts for LTV systems by distinguishing uniform stability from non-uniform stability which are related with the initial time of the system, making the stability analysis of LTV systems much more complicated than the time-invariant ones (see Kalman & Bertram, 1960 for a comprehensive introduction of different stability concepts for dynamics systems and the relationship among them). In the literature there are a few Lyapunov stability theorems for testing uniformly asymptotic stability (namely, uniformly exponential stability) (see Anderson & Moore, 1969, Kalman & Bertram, 1960, Ramarajan, 1986, Rugh, 1996 and the references therein) while few results are available for testing non-uniformly asymptotic stability.

In consideration of the complexity of stability analysis of LTV systems, considerable research effort has been taken to find less conservative *sufficient* conditions. One type of the efficient methods among them relies on the use of Lyapunov function by approximating the LTV systems by time-invariant ones (Ilchmann, Owens, & Pratzel-Wolters, 1987; Mullhaupt et al., 2007). Another type of methods adopts the ideas found in robust control theory by

☆ This work was supported in part by the National Natural Science Foundation of China under Grant Numbers 61273028, 61322305 and 61573120, by the Natural Science Foundation of Heilongjiang Province of China under Grant Number F2015007, and by the Foundation for the Author of National Excellent Doctoral Dissertation of China under Grant 201343. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Akira Kojima under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2015.12.030

modeling the time-varying elements as uncertainties imposed on the linear time-invariant plants so that the theory built for linear time-invariant systems can be adopted (see, for example, Cao & Lam, 2000, Petersen, 1985, Petersen, 1988 and Xu, Lam, & Zou, 2009). Besides, the Bellman–Gronwall inequality approach (which is also known as perturbation analysis) is also proven to be very effective in deriving non-conservative conditions for guaranteeing asymptotic stability for LTV systems (Bellman, 1953; Solo, 1994).

The asymptotic stability test of LTV system has been listed as the FIRST open problem in mathematical systems and control theory (see Aeyels & Peuteman, 1999). It is mentioned in Aeyels and Peuteman (1999) that this problem is non-trivial and some specific versions of this problem were even examined with the framework of complexity theory (Aeyels & Peuteman, 1999). It is further commented in Aeyels and Peuteman (1999) that: "The study of the stability of a system $\dot{x}(t) = A(t)x(t)$ with a system matrix $A(t)$ which is neither fast time-varying nor slowly time-varying is an ambitious task". Hence, by considering that the stability test of LTV systems is extremely difficult and has not yet been fully developed, there is still a need to make an effort on this issue. In this paper motivated by Rugh (1996), Huang, Hollot, and Xu (1991) and the comparison principle Khalil (2002), we make some new observations on the stability analysis of LTV systems. With the help of the notion of stable functions, a series of differential Lyapunov inequality (DLI) based necessary and sufficient conditions is derived to test asymptotic stability, (non-uniformly) exponential stability and uniformly exponential stability. The main feature of these criteria is that they share the same DLI and do not require the right hand side of the DLI be negative all the time. The (non-uniformly) exponential stability concept is then utilized to study the asymptotic stability of a class of triangular LTV systems. A couple of examples with some of them borrowed from the literature is carried out to demonstrate the effectiveness of the obtained results.

The remainder of this paper is organized as follows. In Section 2, we recall the different stability concepts and the corresponding criteria based on the state transition matrix. The main results are provided in Section 3 in which the concept of state functions is introduced and studied in Section 3.1, DLIs based criteria are introduced in Section 3.2 and stability of a class of upper-triangular LTV systems is investigated in Section 3.3. Some examples are provided in Section 4 to illustrate the obtained results. Finally, Section 5 concludes this paper.

2. Stability concepts of LTV systems

Throughout this paper, if not specified, we let $J = [t^*, \infty)$ with t^* being some finite number. We use $\mathbb{C}^1(J, \Omega)$ and $\mathbb{PC}(J, \Omega)$ to denote respectively the space of Ω-valued continuously differentiable functions and piecewise continuous functions defined on J. The acronym WGDR refers to "with guaranteed decay rate" and $|\cdot|$ refers to the usual Euclidean norm. Consider the following linear time-varying (LTV) system

$$\dot{x}(t) = A(t)x(t), \quad t \in J, \tag{1}$$

where $A(t) \in \mathbb{PC}\left(J, \mathbf{R}^{n\times n}\right)$. Denote the state transition matrix for this system as $\Phi(t, t_0),\ \forall t, t_0 \in J, t \geq t_0$.

Definition 1. The LTV system (1) is said to be

(1) stable if for any $\varepsilon > 0$ and for any $t_0 \in J$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x(t_0)| \in [0, \delta] \Rightarrow |x(t)| \in [0, \varepsilon],\ \forall t, t_0 \in J, t \geq t_0$;
(2) uniformly stable if δ in Item 1 is independent of t_0;
(3) asymptotically stable (AS) if it is stable and, for any $t_0 \in J$ and any $\varepsilon > 0$, there exist two positive scalars $\eta = \eta(t_0)$ and $T = T(t_0, \varepsilon)$ such that $|x(t_0)| \in [0, \eta] \Rightarrow |x(t)| \in [0, \varepsilon],\ \forall t \geq T + t_0$;
(4) uniformly asymptotically stable (UAS) if it is uniformly stable and η and T in Item 3 are independent of t_0;
(5) exponentially stable (ES) [WGDR α with $\alpha > 0$] if, for any given $t_0 \in J$, there exist two scalars $k(t_0) > 0$ such that

$$|x(t)| \leq k(t_0)\,|x(t_0)|\,\mathrm{e}^{-\alpha(t-t_0)}, \quad \forall t, t_0 \in J, t \geq t_0; \tag{2}$$

(6) uniformly exponentially stable (UES) [WGDR α] if $k(t_0)$ in the above item is independent of t_0.

Definition 1 except for Item 5 is recalled from Harris and Miles (1980). It is well-known that the stability of an LTV system is completely characterized by its state transition matrix, as recalled in the following lemma.

Lemma 1. *The LTV system (1) is*

(1) *stable if and only if, for any $t_0 \in J$, there exists a $k(t_0) > 0$ such that (Theorem 5.1 in Harris & Miles, 1980)*

$$|\Phi(t, t_0)| \leq k(t_0), \quad \forall t, t_0 \in J, t \geq t_0; \tag{3}$$

(2) *uniformly stable if and only if $k(t_0)$ in (3) is independent of t_0 (Theorem 6.4 in Rugh, 1996);*
(3) *AS if and only if (3) is satisfied and (Theorem 5.2 in Harris & Miles, 1980)*

$$\lim_{t\to\infty} |\Phi(t, t_0)| = 0; \tag{4}$$

(4) *ES [WGDR α] if and only if, for any $t_0 \in J$, there exists a scalar $k(t_0) > 0$ such that*

$$|\Phi(t, t_0)| \leq k(t_0)\mathrm{e}^{-\alpha(t-t_0)}, \quad \forall t, t_0 \in J, t \geq t_0; \tag{5}$$

(5) *UES [WGDR α] if and only if $k(t_0)$ in (5) is independent of t_0 (Theorem 6.7 in Rugh, 1996);*
(6) *UAS if and only if it is UES (Theorem 6.13 in Rugh, 1996).*

Since the notion of the ES (2) is not well recognized in the literature (we noticed that a slightly different definition of ES was given in Anderson, Ilchmann, and Wirth (2013)), Item 4 of Lemma 1 seems not available in the literature; yet its proof is simple and can be carried out by combining the proofs for Items 2 and 5 (see Rugh, 1996).

Remark 1. It follows from Items 5–6 of Lemma 1 that the LTV system (1) is UES only if $|\Phi(t_0 + T, t_0)|,\ t_0 \in J$ and $|\Phi(t_0\delta, t_0)|,\ t_0 > 0$ are uniformly (with respect to t_0) bounded for any given $T > 0$ and $\delta > 1$, namely, if there exists a $T > 0$ or a $\delta > 1$ such that $\lim_{t_0\to\infty} |\Phi(t_0 + T, t_0)| = \infty$ or $\lim_{t_0\to\infty} |\Phi(t_0\delta, t_0)| = \infty$, then the LTV system (1) is not UES.

It follows from Lemma 1 that "UAS" $\Leftrightarrow$ "UES" $\Longrightarrow$ "ES" $\Longrightarrow$ "AS" $\Longrightarrow$ "Stable" and "UAS" $\Longrightarrow$ "Uniformly Stable" $\Longrightarrow$ "Stable". We provide some examples to demonstrate that the converse of the above relationships is not true. The first system is AS but is not ES.

Example 1. Consider the following scalar LTV system (see p. 105 in Rugh, 1996)

$$\dot{x}(t) = A(t)x(t), \quad A(t) = -\frac{2t}{1+t^2},\ \forall t \in J = [0, \infty).$$

This system is not UES since (see p. 105 in Rugh, 1996)

$$\Phi(t, t_0) = \frac{1+t_0^2}{1+t^2}. \tag{6}$$

By a similar approach as in Rugh (1996) we can show that it is also not ES.

The next system is ES but is not UES.