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Brief paper

A refinement of Matrosov's theorem for differential inclusions☆

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ABSTRACT

This note presents a refinement of Matrosov's theorem for a class of differential inclusions whose set-valued map is defined as a closed convex hull of finitely many vector fields. This class of systems may arise in the analysis of switched nonlinear systems when stability with arbitrary switching between the given vector fields is considered. Assuming uniform global stability of a compact set, it is shown that uniform global attractivity of the set can be verified by tailoring Matrosov functions to individual vector fields. This refinement of Matrosov's theorem is an extension of the existing Matrosov results which may be easier to apply to certain differential inclusions than existing results, as demonstrated by an example.

1. Introduction

Matrosov's theorem allows one to check uniform asymptotic stability of time-varying systems in situations when uniform stability has already been established (Matrosov, 1963). In its original form (Matrosov, 1963), Matrosov's theorem establishes convergence via a positive definite Lyapunov function whose derivative is negative semi-definite and an auxiliary not necessarily sign-definite (Matrosov) function whose derivative is strictly negative on the neighbourhood of the set where the derivative of the Lyapunov function is equal to zero. A simpler version of Matrosov's theorem was presented in Paden and Panja (1988). Inspired by Paden and Panja (1988), a generalisation of this theorem with multiple auxiliary Matrosov functions was first proposed in Loria, Panteley, Popović, and Teel (2005). A Matrosov result for differential inclusions can be found in Teel, Loria, and Panteley (2002) but this result uses only two Matrosov functions. Results in Sanfelice and Teel (2009) apply to a class of differential inclusions but those results do not have the added flexibility of the current work, as it will be demonstrated via an example. As pointed out in Sanfelice and Teel (2009), Matrosov's theorem provides alternative conditions to invariance principles for asymptotic stability of time-invariant systems (Krasovskii, 1963; LaSalle, 1967).

Matrosov's theorem was extended to discrete-time systems (Teel, Nešić, Loria, & Panteley, 2010), parameterised discrete-time systems (Nešić & Teel, 2004), hybrid systems (Sanfelice & Teel, 2009) and stochastic systems in Teel (2013a,b). Various Lyapunov function constructions via Matrosov functions were presented in Malisoff and Mazenc (2007), Mazenc, Malisoff, and Bernard (2009) and Mazenc and Nešić (2007) for continuous-time systems.

The main purpose of this paper is to present a refinement for Matrosov's theorem for a class of differential inclusions that may arise in the stability analysis of switched nonlinear systems. The set-valued map defining the considered class of inclusions is generated as a closed convex hull of a finite set of time-varying vector fields. We assume that a compact set $\mathcal{A}$ is uniformly globally stable for the inclusion and we prove uniform global attractivity of the set via the Matrosov functions. It turns out that constructing Matrosov functions directly for the inclusion is in general difficult. We show that if we construct functions that are Matrosov functions for each of the constituting vector fields, then this implies uniform global attractivity of the set $\mathcal{A}$; this can be much simpler than finding Matrosov functions directly for the inclusion as illustrated by an example. We illustrate our result by an example with two vector fields.

The paper is organised as follows. We first present Matrosov's Theorem for general differential inclusions that generalises results

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in Loria et al. (2005) and it is essentially contained in Sanfelice and Teel (2009, Theorem 4.1). Compared to the results in Teel et al. (2002) it allows for more than two functions and it uses a simplified condition from Paden and Panja (1988). A refinement of Matrosov's Theorem for a class of differential inclusions is given in Section 3; this is the main result of our paper. We illustrate our results by applying our main result to an example taken from Lee, Tan, and Nešić (2015). Summary is given in the last section.

2. Matrosov theorem for differential inclusions

In this section, we consider differential inclusions of the form:

$$\dot{x} \in F(t, x), \tag{1}$$

where $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. A solution of a differential inclusion, defined by a set-valued mapping F (see Eq. (1)), is a locally absolutely continuous $\mathbb{R}^n$-valued function defined on some interval of the form $[a, b)$, with $0 \leq a < b$ (where b could be the infinity), such that the derivative of x at the time instant t is in the set $F(t, x(t))$ for almost all t in $[a, b)$. The interval $[a, b)$ is defined as $Dom(x)$ and a is denoted as $t_0(x)$. The solution x is said to be complete if $b = \infty$, and it is said to be maximal if there is no other solution y such that $Dom(x)$ is contained in $Dom(y)$ and $x(t) = y(t)$, $\forall t \in Dom(x)$.

For a set-valued function $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, let $D(F) := \{x \in \mathbb{R}^n \,|F(t, x) \neq \emptyset, \forall t \geq 0\}$. The following assumption can be used to guarantee the existence of solutions of the system (1) from points $x \in D(F)$ (see Filippov, 1988 for the existence of solutions and more detailed discussion in Goebel, Sanfelice, & Teel, 2009).

Assumption 1. A set-valued function $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following conditions:

(a) $D(F)$ is a nonempty open subset of $\mathbb{R}^n$.
(b) F is outer semi-continuous[1] on $\mathbb{R}_{\geq 0} \times D(F)$.
(c) For each $x \in D(F)$ and each $t \geq 0$, $F(t, x)$ is compact and convex.
(d) F is locally bounded on $\mathbb{R}_{\geq 0} \times D(F)$.[2]

We show that an extension of Matrosov's theorem for differential equation presented in Loria et al. (2005) is easy to state for differential inclusions of the form (1), see Theorem 1. The set of solutions for this inclusion starting from an initial time t_0 with an initial condition $x_0 = x(t_0)$ is denoted as $\mathcal{S}(t_0, x_0)$. Let $|\xi|$ denote the Euclidean norm of a vector $\xi \in \mathbb{R}^n$. Given a set $\mathcal{A} \subset \mathbb{R}^n$, we use $|\xi|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |\xi - z|$ to denote the distance of $\xi \in \mathbb{R}^n$ from $\mathcal{A}$.

Definition 1. We use the following stability notions for the differential inclusion (1):

- A compact set $\mathcal{A}$ is said to be uniformly stable (US) if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that we have that $|x_0|_{\mathcal{A}} \leq \delta$ implies that $|x(t)|_{\mathcal{A}} \leq \epsilon$ for all $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n, x \in \mathcal{S}(t_0, x_0)$ and $t \in Dom(x)$.
- $\mathcal{A}$ is said to be uniformly globally stable (UGS) if it is US and the solutions are uniformly globally bounded with respect to $\mathcal{A}$; that is, for any $\Delta > 0$ there exists $M > 0$ such that $|x_0|_{\mathcal{A}} \leq \Delta$ implies that $|x(t)|_{\mathcal{A}} \leq M$ for all $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n, x \in \mathcal{S}(t_0, x_0)$ and $t \in Dom(x)$.
- The set $\mathcal{A}$ is uniformly globally attractive (UGA) if for any strictly positive Δ, ϵ, there exists $T = T(\Delta, \epsilon)$ such that $|x_0|_{\mathcal{A}} \leq \Delta$ implies that $|x(t)|_{\mathcal{A}} \leq \epsilon$ for all $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n, x \in \mathcal{S}(t_0, x_0)$ and $t \in Dom(x)$ with $t \geq t_0 + T$.
- The set $\mathcal{A}$ is uniformly globally asymptotically stable if it is UGS and UGA. ■

Remark 1. It is worthwhile to highlight that in the definitions of stability properties, it is not assumed that solutions are complete, see a similar discussion in Sanfelice and Teel (2009).

To state Theorem 1, we need to introduce some notation. Let $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the strictly positive real numbers (δ, Δ) and the compact set $\mathcal{A} \subset \mathbb{R}^n$ be given. Define $D = D(\delta, \Delta) := \{\xi \in \mathbb{R}^n : |\xi|_{\mathcal{A}} \in [\delta, \Delta]\}$. Let $\gamma > 0$ and $\mathbb{B}^n := \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$; then $\gamma\mathbb{B}^n = \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma\}$.

Next, the Matrosov property and Matrosov function are defined.

Definition 2 (*Matrosov Property*)**.** A finite set of continuous functions $\{Y_j\}_{j=1}^r$, $Y_j : \gamma\mathbb{B}^m \times D \to \mathbb{R}$ for each $j \in \{1, \ldots, r\}$, is said to have *the Matrosov property relative to* (γ, D) if, with the additional definitions $Y_0 \equiv 0$ and $Y_{r+1} \equiv 1$, we have the following property:

For each $j \in \{0, \ldots, r\}$, if $(z, x) \in \gamma\mathbb{B}^m \times D$ and $Y_0(z, x) = \cdots = Y_j(z, x) = 0$ then $Y_{j+1}(z, x) \leq 0$.

Remark 2. Due to $Y_0 \equiv 0$, this property with $j = 0$ implies that $Y_1(z, x) \leq 0$ for all $(z, x) \in \gamma\mathbb{B}^m \times D$; due to $Y_{r+1} \equiv 1$, the property with $j = r$ implies that there are no points $(z, x) \in \gamma\mathbb{B}^m \times D$ for which $Y_1(z, x) = \cdots = Y_r(z, x) = 0$.

Definition 3. A finite set of continuously differentiable functions $\{W_j\}_{j=1}^r$, where $W_j : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ for each $j \in \{1, \ldots, r\}$, are said to be *Matrosov functions for* (F, D) if there exists a function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$, a positive real number γ, and continuous functions $Y_j : \gamma\mathbb{R}^m \times D \to \mathbb{R}, j \in \{1, \ldots, r\}$ that have the Matrosov property relative to (γ, D) and, for each $j \in \{1, \ldots, r\}$,

$$\max\left\{|W_j(t, x)|, |\phi(t, x)|\right\} \leq \gamma \qquad \forall (t, x) \in \mathbb{R}_{\geq 0} \times D \tag{2a}$$

$$\nabla_t W_j(t, x) + \langle \nabla_x W_j(t, x), \xi\rangle \leq Y_j(\phi(t, x), x) \quad \forall (t, x) \in \mathbb{R}_{\geq 0} \times D, \xi \in F(t, x). \tag{2b}$$

Now we can state the main result of this section. This result is a direct generalisation of Theorem 1 in Loria et al. (2005) and it is essentially contained in Sanfelice and Teel (2009, Theorem 4.1) and so its proof is omitted.

Theorem 1. *For the differential inclusion (1), if the compact set $\mathcal{A}$ is UGS and, for each pair of strictly positive real numbers (δ, Δ), there exist Matrosov functions for (F, D), then $\mathcal{A}$ is UGAS.* ■

3. A refinement of Matrosov's theorem for differential inclusions

In this section, we consider differential inclusions (1) where the set-valued mapping F can be written as

$$F(t, x) = \mathrm{co}\left(\bigcup_{i=1}^{m_f} f_i(t, x)\right), \quad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \tag{3}$$

for a given positive integer m and functions $f_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, \ldots, m_f\}$.

[1] For a set-valued function $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, it is said to be outer semicontinuous on $\mathbb{R}_{\geq 0} \times D(F)$ if for any $(t, x) \in \mathbb{R}_{\geq 0} \times D(F)$ and any sequence $\{(t_n, x_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}_{\geq 0} \times D(F)$ with $(t_n, x_n) \to (t, x)$ as $n \to \infty$, and $y_n \in F(t_n, x_n) \to y$ as $n \to \infty$, then $y \in F(t, x)$.

[2] For a set-valued function $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, it is said to be locally bounded on $\mathbb{R}_{\geq 0} \times D(F)$ if for any $(t, x) \in \mathbb{R}_{\geq 0} \times D(F)$, there exist a $\delta(t, x) > 0$ and a compact set $K(t, x) \subset \mathbb{R}^n$ such that $F(s, y) \subset K$, $\forall |s - t| < \delta$, $|y - x| < \delta$.