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Characterization of best Chebychev approximation using the frequency response of IIR digital filters with convex stability

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ABSTRACT

This paper deals with the application of Chebychev's approximation theory to IIR digital filter frequency response (FR) approximation. It explores the properties of the frequency response of IIR digital filters as a nonlinear complex approximating function; IIR digital filter frequency response is used to approximate a prescribed magnitude and phase responses. The approximation problem is closely related to optimization. If the set of approximating functions is non-convex, the optimization problem is difficult and may converge to a local minimum. The main results presented in the paper are proposing a convex stability domain by introducing a condition termed "sign condition" and characterization of the best approximation by the Global Kolmogorov's Criterion (GKC). The Global Kolmogorov's Criteria is shown to be also a necessary condition for the approximation problem. Finally, it is proved that the best approximation is a global minimum. The sign condition can be incorporated as a constraint in an optimization algorithm.

1. Introduction

The transfer function of an IIR digital filter is

$$H(z) = \frac{N(z^{-1})}{D(z^{-1})} = \frac{\sum_{k=0}^{m} b_k z^{-k}}{1 + \sum_{l=1}^{n} a_l z^{-l}},$$

$$z \in \mathbf{U},\ U \text{ is the unit disc.} \tag{1}$$

Its frequency response is $H(\omega) = H(z)|_{z=e^{j\omega}}$. This function is used to approximate a prescribed frequency response on a compact interval, Ω.

In many applications of digital signal processing filter design with arbitrary magnitude and phase responses is required.

One design trend of IIR digital filters is to meet frequency response magnitude specifications that minimize a specific error norm (L_∞ norm). The designed filter may have a nonlinear phase. An all-pass filter is cascaded with the filter as an equalizer [1,2]. The equalizer is a real nonlinear phase function of the all-pass filter [3]. The minimum error is often characterized by the alternation theorem (equiripple of the error on a frequency interval).

One of the drawbacks associated with the use of equalizer is that the number of independent coefficients in an all-pass section is less than the number of the filter coefficients. Moreover, based on approximation theory, the original coefficients of the IIR digital filter, a and b, are no longer the independent coefficients for the magnitude approximation problem. The valid independent approximation parameters are, in this case, the coefficients of the magnitude which are functions of a and b that are probably not easily solved.

Another trend is to approximate both magnitude and phase simultaneously using the complex FR functions. The major challenges in any approximation problem are: existence, uniqueness, characterization of best approximation and designing an algorithm. The Chebychev approximation with general continuous complex valued rational functions is tackled in [34–36]. As it was established by Walsh [34], the existence of best approximation is guaranteed provided the domain of approximation is compact and has no singularity points. In addition, the best approximation is known to be non-unique [36].

In the real approximation the alternation theorem is the tool for characterizing the best approximation. This theorem no longer holds in the complex case. The main tools for characterization of an optimal solution in the complex case are the Global Kolmogorov Criterion (GKC) and the Local Kolmogorov Criterion (LKC) [30,31]. GKC is generally a sufficient condition while LKC is a necessary condition. The intimate connection between approximation and optimization is well recognized [32,33]. The optimization algorithm is used to determine the coefficients of a stable IIR digital filter that minimizes the max-norm error (L_∞). Various design methods are proposed to compute an optimal solution [11–22]. The optimization problem is difficult if the set of approximating functions are non-convex. In such cases, the algorithm may converge to a local minimum. Another major problem in the design of IIR

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digital filters is stability. Some design methods start with a point corresponding to a stable IIR digital filter, i.e., the roots of $D(A,z)$ lie inside the unit disc (Schur Polynomial), and monitor the stability iteratively; other methods follow stabilization steps. Numerical optimization algorithms incorporate the stability requirements in a constrained optimization setting. A linear programming algorithm was proposed for the optimization utilizing the positive realness of $D(A,z)$, $(Re\{D(A,\omega)\}>0)$, to ensure stability [13]. The convex stability of $(N(z), D(z))$ and the positive realness of $H(z)$ are utilized to obtain a convex set of IIR digital filters [21]. Rouche's condition on the denominator perturbation is incorporated to preserve stability [20]. Stability margin approach was proposed in [18]. Iterative Lyaponov inequality constraint is incorporated for the filter stability [17].

The objective of this paper is to investigate the approximation properties of the rational complex FR functions using non-linear Chebychev approximation theory. The concepts of "functions with betweeness property" [25] and theory of "regular systems" [27] play an important role in this paper. The results of this study are three folds: it proposes a convex stability domain of FR functions by introducing a condition termed "*sign condition*". In addition, the proof that GKC (Theorem 2) is also a necessary condition in order to characterize a best approximation, is provided. Finally, the best approximation is shown to be a global minimum (Theorem 3). This sign condition has to be incorporated as a constraint in the optimization algorithm.

The interested reader about convex stability is invited to review the references [4, Chapter 7] [5–8] (see also Appendix A).

This paper is organized as follows: some definitions are provided in Section 2. Section 3 states the complex approximation problem of FR functions. The results of the paper are included in Section 4. Section 5 presents examples. The conclusion is presented in Section 6. Appendix A is about convex stability.

2. Preliminaries

2.1. IIR digital filters

The transfer function of an IIR digital filter is defined in Eq. (1) where $N(z)$ and $D(z)$ are relatively prime of fixed degrees m and n with cardinality $m \leqslant n$. The sets of parameters $\{A=(a_1\dots a_n),\ a_0=1\}$ and $\{B=(b_0\dots b_m)\}$ are real.

The set of FR functions of stable IIR digital filters is denoted by $\mathcal{H}$.

A digital IIR digital filter is stable if the denominator $D(A,z)$ has all its zeros inside the unit disc., i.e. $D(A,z)$, is a Schur polynomial. $D(A,\omega)\neq 0$ on the boundary of the unit circle and $Re[D(A,\omega)]>0$ [13].

The convex combination of two polynomials, $D_0(\omega)$ and $D_1(\omega)$, is

$$D_\lambda = D_0 + \lambda(D_1 - D0),$$

where λ belongs to $[0,1]$.

The real and imaginary parts of $D(A,\omega)$ are respectively,

$$g(A,\omega) = 1 + \sum_{l=1}^{n} a_l \cos(l\omega), \tag{2}$$

$$u(A,\omega) = \sum_{l=1}^{n} a_l \sin(l\omega). \tag{3}$$

3. Statement of the Chebychev approximation problem

The following brief review considers the general problem of approximation of a continuous function, $f(\omega)$, by an approximating function depending on a finite number of parameters. Thus, the problem under consideration is to approximate, $f(\omega)$, by an approximating function, $F(A,\omega)$, which may depend on the parameter, A, in a linear or non linear way. The problem is to determine those parameters, A^*, which make, $F(A^*,\omega)$, closest to $f(\omega)$ with respect to some norm, i.e., $\|f(\omega)-F(A^*,\omega)|$ is a minimum. The functions, $f(\omega)$, and, $F(A,\omega)$, can be real or complex functions. The function, $F(A^*,\omega)$, may be termed best approximation, optimal approximation and minimal solution. Once the problem is formulated in a mathematical form, there are four main issues related to its solution after the choice of $F(A,\omega)$: existence, uniqueness, characterization and computation of $F(A^*,\omega)$. A norm is defined by $\|f\|_p = (\int |(f(\omega)|^p\, d\omega))^{\frac{1}{p}}$ and denoted L_p norm. The norms L_1, L_2, and L_∞ are often used in the approximation theory.

3.1. Mathematical formulation of the approximation problem

Let $C(\Omega)$ be the space of continuous complex valued function on a real compact interval, Ω, endowed with the max-norm, L_∞

$$\big\|H(A,B,.)\big\| = \max_{\omega}\big|H(A,B,\omega)\big|. \tag{4}$$

Let $H_d(\omega)\in C(\Omega)\setminus\mathcal{H}$ be a prescribed frequency response and $H(A,B,\omega)\in\mathcal{H}$ be the approximating function. For example, $H_d(\omega)$ may be $\Gamma e^{-j\phi(\omega)}\in C(\Omega)\backslash\mathcal{H}$, where Γ is a constant and $\phi(\omega)$ is a linear function of ω. The error function of approximation is defined as:

$$e = H_d(\omega) - H(A,B,\omega). \tag{5}$$

This function attains its norm on a discrete point set $M^*\subset\Omega$ with cardinality $\geqslant m+n+2$ points. The minimum solution of the Chebychev approximation problem $H_0(a^*,b^*,\omega)$ is the solution of:

$$\begin{aligned} E^* &= \big\|H_d(.) - H_0\big(A^*,B^*,.\big)\big\| \\ &= \min_{(A,B)}\max_{\omega}\big|H_d(\omega) - H(A,B,\omega)\big|, \end{aligned} \tag{6}$$

where E^* is the max-norm of e. This solution is characterized by the GKC and the LKC [30].

4. Results

The keys of this study are the concept of "betweeness property" [24,25] and the more general concept "regular systems" [27,28]. These two concepts were introduced in the context of nonlinear approximation theory as a generalization of convexity. A variant concept termed "weak betweeness property" has been applied for characterization and uniqueness of best approximation [29]. The three concepts have considered complex rational functions provided that the denominator is a real positive function.

The convex stability and the line homotopy [37] are additional concepts playing an important role in the study.

In the approximation problem at hand, the concepts of betweeness property and regular systems have been applied to the complex FR functions provided the denominator, $D(A,\omega)$, is a complex function and non-zero on the unit circle.

4.1. The main result of the paper is Theorem 1

Its consequences are proposing a convex stability domain in FR functions by introducing a condition, i.e., "sign condition". Recalling, from introduction that GKC is generally a sufficient condition in order to characterize the best approximation; the investigation of this work shows that GKC is also a necessary condition (Theorem 2) under the existence of the monotone sequence denoted h_λ