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On primitivity of sets of matrices[☆]

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ABSTRACT

A nonnegative matrix A is called primitive if A^k is positive for some integer $k > 0$. A generalization by Protasov and Voynov (2012) of this concept to finite sets of matrices is as follows: a set of matrices $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$ is primitive if $A_{i_1}A_{i_2}\ldots A_{i_k}$ is positive for some indices $i_1, i_2, \ldots, i_k$. The concept of primitive sets of matrices comes up in a number of problems within the study of discrete-time switched systems. In this paper, we analyze the computational complexity of deciding if a given set of matrices is primitive and we derive bounds on the length of the shortest positive product.

We show that while primitivity is algorithmically decidable, unless $P = NP$ it is not possible to decide primitivity of a matrix set in polynomial time. Moreover, we show that the length of the shortest positive sequence can be superpolynomial in the dimension of the matrices. On the other hand, defining $\mathcal{P}$ to be the set of matrices with no zero rows or columns, we give a combinatorial proof of the Protasov–Voynov characterization (2012) of primitivity for matrices in $\mathcal{P}$ which can be tested in polynomial time. This latter observation is related to the well-known 1964 conjecture of Černý on synchronizing automata; in fact, any bound on the minimal length of a synchronizing word for synchronizing automata immediately translates into a bound on the length of the shortest positive product of a primitive set of matrices in $\mathcal{P}$. In particular, any primitive set of $n \times n$ matrices in $\mathcal{P}$ has a positive product of length $O(n^3)$.

1. Introduction

A $n \times n$ matrix A which is entrywise nonnegative is said to be primitive if every entry of A^k is positive for some positive integer k. It is well-known (see Horn & Johnson, 1995, Corollary 8.5.9) that this is the case if and only if $A^{n^2-2n+2} > 0$ so the primitivity of a matrix is easy to verify algorithmically. The Protasov–Voynov primitivity generalizes this notion to sets of matrices (Protasov & Voynov, 2012): a finite set of m nonnegative matrices $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$ is primitive if $A_{i_1}A_{i_2}\ldots A_{i_k}$ is (entrywise) positive for some indices $i_1, i_2, \ldots, i_k \in \{1, \ldots, m\}$.

The property of Protasov–Voynov primitivity of a set of matrices is important in several applications. In particular, its presence enables one to use efficient algorithms for the computation of the Lyapunov exponent of a stochastic switching system (we refer the reader to Jungers, 2009; Liberzon, 2003; Shorten, Wirth, Mason, Wulff, & King, 2007 for a general introduction to switching systems). Given a finite set of matrices $\mathcal{M} \subset \mathbb{R}^{n\times n}$, one can define a stochastic switched system as:

$$x_{k+1} = A_{i_k}x_k, \quad A_{i_k} \in \mathcal{M}, \tag{1}$$

where for simplicity let us make the assumption that each A_{i_k} is chosen randomly from the uniform distribution on $\mathcal{M}$. Such models are commonly used throughout stochastic control; for example, they are a common choice for modeling manufacturing systems with random component failures (see Boukas, 2005, Chapter 1). The *Lyapunov exponent* of this system is defined by the following limit (where $\mathcal{E}$ denotes the expectation):

$$\lambda = \lim_{k\to\infty} \frac{1}{k}\, \mathcal{E} \log \left\| A_{d_k} \cdots A_{d_1} \right\|. \tag{2}$$

The Lyapunov exponent characterizes the rate of growth of the switching system with probability one. While it is hard to compute in general, it turns out that *in the particular case of primitive sets of matrices, efficient algorithms are available.* We refer the reader to Protasov (2010a,b) and Protasov and Jungers (2013a,b) for the algorithms.

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Second, the concept of primitivity is also related to the so-called consensus problem. Here the matrices in $\mathcal{M}$ are further taken to be stochastic matrices and the question is whether the recursion of Eq. (1) almost surely converges to $\alpha\mathbf{1}$, i.e., to a multiple of the all-ones vector. In this case, we say that the iteration achieves consensus on the value α. Such "consensus iterations" appear in a number of applications, and there is now a considerable literature on the consensus problems providing necessary or sufficient conditions under various assumptions on the switching—we refer the reader to the classical and modern papers (Blondel & Olshevsky, 2014; Chatterjee & Seneta, 1977; DeGroot, 1974; Jadbabaie, Lin, & Morse, 2003; Liu, Mou, Morse, Anderson, & Yu, 2011; Lorenz & Lorenz, 2010; Tsitsiklis, Bertsekas, & Athans, 1986) for examples of such conditions and discussions of applications. The consensus problem naturally leads to the concept of primitivity, since a positive entry in a product represents an interaction between the two corresponding agents (that is, the two corresponding entries of x_0).

Finally, the problem of matrix primitivity is perhaps the simplest possible reachability problem for switched systems: given an unknown initial state in the nonnegative orthant, can we choose at each step a matrix A_{i_k} from the set of nonnegative matrices $\{A_1, \ldots, A_m\}$ so that the final state of Eq. (1) is in the interior of the nonnegative orthant? As we show in this paper, even this simple and stylized reachability problem faces significant computational obstructions.

In this paper, we study the problem of recognizing primitivity and related problems. Given a set of $n \times n$ nonnegative matrices $\mathcal{M} = \{A_1, \ldots, A_m\}$ one would like to determine, efficiently if possible, whether or not $\mathcal{M}$ is primitive. This is closely related to the problem of bounding the length of the shortest positive product of matrices from $\mathcal{M}$, which we denote by $l(\mathcal{M})$. Indeed, an upper bound on $l(\mathcal{M})$ immediately translates into algorithms for checking primitivity by simply checking every possible product of length smaller or equal to this bound (though in some particular cases more efficient algorithms can be used (see Section 3)).

1.1. Our results

This paper is consequently concerned with upper bounds on the length of $l(\mathcal{M})$ as well as algorithms and complexity of verifying existence of a positive product of matrices taken in a given set $\mathcal{M}$. Our main results are:

(1) We show in Section 2 that recognizing primitivity is decidable but NP-hard as soon as there are three matrices in the set. Primitivity can be decided in polynomial time for one matrix and so we leave the computational complexity of the case of two matrices unresolved.
(2) We also show in Section 2 that the shortest positive product may have a length that is superpolynomial in the dimension of the matrices, even with a fixed number of matrices in the set.
(3) We consider in Section 3 the primitivity problem under the additional mild assumption that all the matrices in the set $\mathcal{P}$ have no zero rows or columns. We provide a combinatorial proof of a previously-known primitivity criterion under this assumption. This resolves an open question of Protasov and Voynov (2012), who first proved the validity of the same criterion using linear algebraic tools, and showed that it can be checked in polynomial time.
(4) We also prove in Section 3 that for primitive sets of matrices in $\mathcal{P}$, the shortest positive product has length $O(n^3)$. Moreover, we show that in this case the length of the shortest positive product is related to the well-known (and unresolved) conjecture of Černý on synchronizing automata. In particular, we show that resolution of the Černý conjecture would improve the above bound to $O(n^2)$. Moreover, any upper bound on the length of the shortest synchronizing word for a synchronizing automaton immediately translates into a bound on the length of the shortest positive product of a set of primitive matrices in $\mathcal{P}$.

1.2. Implications of our results

Our results have implications for a number of ongoing research efforts within the field of discrete-time switched systems. First, they complement previous results from Protasov (2010a,b) and Protasov and Jungers (2013a,b) which provided simple algorithms for the computation of Lyapunov exponents of nonnegative matrices from $\mathcal{P}$ for which a positive product exists. If the existence of a positive product is not guaranteed, then the above papers provided more complex and computationally involved protocols relying on quasiconcave maximization. Our results here provide an efficient way of verifying when it is possible to use the lower complexity protocols to compute Lyapunov exponents of matrices from $\mathcal{P}$.

Second, our results shed light on the problem of consensus with randomly chosen matrices at each step. Our results in Section 3 give a necessary and sufficient condition for primitivity of stochastic matrices (corresponding to consensus on a value in the strict convex hull of the initial states) which have no zero rows and columns. To our knowledge, the only previous case when necessary and sufficient conditions for consensus with randomly chosen matrices have been provided has been in Tahbaz-Salehi and Jadbabaie (2008) for the case of matrices with positive diagonals. Since stochastic matrices cannot have a zero row by definition, our results in Section 3 effectively require only the absence of zero columns, significantly expanding the set of stochastic matrices for which necessary and sufficient conditions for random consensus can be given.

Finally, as we previously remarked, matrix primitivity is among the mathematically simplest possible reachability questions one can pose for switched systems. The NP-hardness results of Section 1 show that, unfortunately, even this problem cannot be decided in polynomial time unless $P = NP$. In particular, this implies that any generalization of this simple reachability problem is NP-hard as well.

For example, the problem of steering an unknown initial state of Eq. (1) to the interior of a given polyhedron by picking the appropriate matrix A_{i_k} at each step is NP-hard, even if the initial condition lies on the boundary of the polyhedron. More broadly, our results suggest that reachability problems for discrete-time switched systems can be NP-hard even after a slew of simplifying assumptions on the matrices involved, the structure of the set to be reached, and the initial condition.

1.3. Related work

The concept of primitive matrix families as we study it here was pioneered in the recent paper (Protasov & Voynov, 2012), which extended the classical Perron–Frobenius theory and provided a structure theorem for the primitive matrix sets in $\mathcal{P}$. A consequence of this theorem was that for matrices in $\mathcal{P}$ primitivity can be tested in polynomial time. The question of finding a combinatorial proof was left open.

Other generalizations of the well-studied primitivity of one single matrix to a set of matrices exist in the literature. See for instance Olesky, Shader, and van den Driessche (2002) or Cohen and Sellers (1982), and Protasov (2013) for a recent paper on so-called k-primitivity.

We note that two recent papers, appearing simultaneously in 2013 with the conference version of this paper (Blondel, Jungers, & Olshevsky, 2013), have also tackled items (3) and (4) above