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Continuous-time stochastic consensus: Stochastic approximation and Kalman–Bucy filtering based protocols☆

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ABSTRACT

This paper investigates the continuous-time multi-agent consensus with stochastic communication noises. Each agent can only use its own and neighbors' information corrupted by random noises to design its control input. To attenuate the communication noises, we consider the stochastic approximation type and the Kalman–Bucy filtering based protocols. By using the tools of stochastic analysis and algebraic theory, the asymptotic properties of these two kinds of protocols are analyzed. Firstly, for the stochastic approximation type protocol, we clarify the relationship between the convergence rate of the consensus error and a representative class of consensus gains in both mean square and probability one. Secondly, we propose Kalman–Bucy filtering based consensus protocols. Each agent uses Kalman–Bucy filters to get asymptotically unbiased estimates of neighbors' states and the control input is designed based on the protocol with precise communication and the certainty equivalence principle. The iterated logarithm law of estimation errors is developed. It is shown that if the communication graph has a spanning tree, then the consensus error is bounded above by $O(t^{-1})$ in mean square and by $O(t^{-1/2}(\log\log t)^{1/2})$ almost surely. Finally, the superiority of the Kalman–Bucy filtering based protocol over the stochastic approximate type protocol is studied both theoretically and numerically.

1. Introduction

In recent years, the distributed coordination of multi-agent systems (MASs) has attracted more and more attention of multi-disciplinary researchers, due to its wide applications in the formation control (Fax & Murray, 2004), the distributed optimization (Nedic & Ozdaglar, 2009), and the flocking problem (Olfati-Saber, 2006). The consensus problem, which is one of the most fundamental topics in the distributed coordination, has been widely studied in the system and control community motivated by Vicsek's model in Vicsek, Czirok, Ben-Jacob, Cohen, and Sochet (1995). Consensus control generally means to design a distributed protocol such that all agents asymptotically reach an agreement on their states. A comprehensive survey on consensus problems can be found in Ren, Beard, and Atkins (2005) and more recent results can be found in Nourian, Caines, and Malhame (2014), Pasqualetti, Borra, and Bullo (2014) and Su and Huang (2012), etc.

Consensus problems with random measurement or communication noises have attracted several researchers since such modeling reflects many practical properties of distributed networks. For the consensus protocol with precise communication, for a given agent, the weighted sum of relative states between neighbors and itself is used to update the agent's state. This weighted sum of relative states can be viewed as a kind of spacial innovation. For the case with communication noises, the spacial innovation is corrupted. To attenuate the noise effect, one idea is using the cautious control, that is, to decrease the algorithm gain. As long as the consensus system evolves, the differences between agents' states become smaller and smaller, then the new information contained in the space innovation corrupted by noises becomes less and less, so a vanishing algorithm gain has to be used. This is so called distributed stochastic approximation type consensus.

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Huang and Manton (2009) considered the discrete-time stochastic approximation type consensus algorithm with fixed topologies. They proved that if the consensus gain $a(k)$ (where k is the discrete time instant) decays with a rate $O(1/k^{\gamma})$, $\gamma \in (0.5, 1]$, the communication graph has the circulant invariance property and strong connectivity, then the algorithm ensures both mean square and almost sure consensus. Huang and Manton (2010) extended the results to the case with general digraphs, which proved that if the digraph contains a spanning tree, then $\sum_{k=0}^{\infty} a(k) = \infty$ and $\sum_{k=0}^{\infty} a^2(k) < \infty$ suffices for both mean square and almost sure consensus. Kar and Moura (2010) considered the discrete-time distributed averaging with quantized data and random link failures. By using dithered quantization, the quantization error sequence is transformed to white noises and the stochastic approximation consensus protocol is employed to ensure mean square convergence of the algorithm. Li and Zhang (2010) considered the case of fixed and time-varying topologies, and they showed if the network switches between jointly-containing-spanning-tree, instantaneously balanced digraphs, then the designed protocol can guarantee that each individual state converges, both almost surely and in mean square, to a common random variable, whose expectation is right the average of the initial states of the whole system. Besides, a rough estimate of the almost sure convergence rate for the consensus error was given. Continuous-time stochastic approximation type consensus problems have also been widely studied. Li and Zhang (2009) showed that if the network is a balanced digraph containing a spanning tree, then a necessary and sufficient condition to guarantee the asymptotic unbiased mean square average-consensus is $\int_0^{\infty} a(t)dt = \infty$ and $\int_0^{\infty} a^2(t)dt < \infty$. More extended results on continuous-time stochastic approximation type consensus protocols can be found for the leader-following cases (Hu & Feng, 2010; Ma, Li, & Zhang, 2010), the case with general digraphs (Wang & Zhang, 2009), the case with time-delay (Liu, Liu, Xie, & Zhang, 2011) and the cases of second-order and linear dynamics with static state feedback (Cheng, Hou, & Tan, 2014; Cheng, Hou, Tan, & Wang, 2011). And recently, this kind of protocols are applied to the containment control of multi-agent systems with random measurement noises (Wang, Cheng, Hou, Tan, & Wang, 2014).

The works on continuous-time stochastic approximation type consensus protocol mainly concentrated on the conditions to ensure the mean square or almost sure consensus. However, its asymptotic convergence rate, which represents the negotiation speed of the agents as time goes to infinity, is rarely investigated in the relevant literature. It is more meaningful to study the relationship among the asymptotic convergence rate, the consensus gain function $a(t)$, and the communication graph. In this paper, motivated by the above discussions, we investigates the asymptotic convergence rate of the continuous-time stochastic approximation type consensus protocol. We consider a representative class of consensus gains which satisfy $\int_0^{\infty} a(t)dt = \infty$ and $\int_0^{\infty} a^2(t)dt < \infty$. Using the basic results of stochastic analysis and algebraic graph theory, in particular the law of the iterated logarithm of stochastic integrals, we get precise estimations of the convergence rate of the consensus error. It is found that if the consensus gain satisfies that $\lim_{t\to\infty}(t^{\gamma}a(t))$ exists and is positive for $\gamma \in (0.5, 1]$, and for $\gamma = 1$, $\lim_{t\to\infty}(ta(t)) > 1/(2\lambda_{\min})$, with $\lambda_{\min}$ denoting the smallest real part of Laplacian eigenvalues of the network graph, we have: (i) the mean square of the consensus error is bounded above by $O(t^{-\gamma})$ asymptotically; (ii) for the case with balanced digraphs, the mean square of the consensus error is bounded both above and below by $\Theta(t^{-\gamma})$ asymptotically; (iii) the consensus error is almost surely bounded above by $O(t^{-\gamma/2+\varepsilon})$, $\forall\, \varepsilon > 0$, asymptotically for the case with undirected graphs. In this paper, we improve the results of Li and Zhang (2009, 2010) in the sense that (i) the mean square convergence rate of the continuous-time stochastic approximation type consensus protocol is first given; (ii) the almost sure convergence rate is estimated more precisely.

Since the vanishing consensus gain function is used in the stochastic approximation type consensus protocol, the communication noises are attenuated at the price of a slower convergence rate of the algorithm (Huang & Manton, 2009). This motivates us to propose another idea to attenuate the communication noises. The received information from neighbors can be filtered firstly to get the estimates of neighbors' states, then the estimates can be used instead of the true states for the control protocol design. This methodology for controller design is often used in single-agent control systems and is called the certainty equivalence principle. It is well-known that the Kalman–Bucy filter is the main tool of state estimation for continuous-time linear systems driven by Gaussian white noises (Kallianpur, 1980; Øksendal, 2010). Here, based on the Kalman-Bucy filtering theory, we design a filter for each noisy communication link to get the asymptotically unbiased estimates of neighbors' states, then the control input of each agent is designed based on the consensus protocol with precise communication (Olfati-Saber & Murray, 2004) and the certainty equivalence principle. We develop the iterated logarithm law of estimation errors and show that if the communication graph has a spanning tree, then this novel Kalman-Bucy filtering based protocol leads to both mean square and almost sure weak consensus. Moreover, the mean square of the consensus error is bounded above by $O(t^{-1})$ asymptotically, and the consensus error for each agent is almost surely bounded above by $O(t^{-1/2}(\log\log t)^{1/2})$. Comparing the convergence rates of these two kinds of protocols, it is shown that the Kalman-Bucy filtering based protocol leads to a higher convergence rate than the stochastic approximation type protocol in some circumstances. Especially, we verify this superiority for the case with undirected graphs.

The paper is organized as follows. In Section 2, we formulate the considered consensus problem. Section 3 gives the asymptotic convergence properties for the stochastic approximation type protocol. In Section 4, we introduce the Kalman-Bucy filtering based consensus protocol and analyze the convergence rate of the consensus error. The protocol is also applied to a leader-following scenario. The asymptotic convergence properties of these two kinds of protocols are compared in Section 5, while a numerical example is presented. Finally, Section 6 concludes the paper and gives some interesting future topics.

In this paper, we will adopt the following notations: $\mathscr{R}^{m\times n}$ denotes the $m \times n$ dimensional real space; $\mathbf{1}_{\mathbf{N}\times\mathbf{1}}$ denotes an $N \times 1$ column vector with all ones; $\mathbf{0}_{\mathbf{N}\times\mathbf{1}}$ denotes an $N \times 1$ column vector with all zeros. For a given vector or matrix A, A^T denotes its transpose, and $\|A\|$ denotes its Frobenius norm. For a given complex number λ, $Re(\lambda)$ denotes its real part and $Im(\lambda)$ denotes its imaginary part. The notion $f(t) = O(g(t))$ denotes $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$; $f(t) = \Omega(g(t))$ denotes $\liminf_{t\to\infty} |f(t)/g(t)| > 0$; $f(t) = \Theta(g(t))$ denotes $0 < \liminf_{t\to\infty} |f(t)/g(t)| \le \limsup_{t\to\infty} |f(t)/g(t)| < \infty$ and $f(t) = o(g(t))$ denotes $\lim_{t\to\infty} |f(t)/g(t)| = 0$.

2. Problem formulation

Let the communication topology of MASs be modeled by a weighted digraph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{A}\}$. The set of nodes $\mathscr{V} = \{1, \ldots, N\}$, and node i represents the ith agent. A pair (j, i) belongs to the edge set $\mathscr{E} \Leftrightarrow$ the jth agent can send information to the ith agent directly. Here, j is called the parent of i, and i is called the child of j. The neighborhood of the ith agent is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} | (j, i) \in \mathscr{E}\}$. Node i is called a source if it has no parent but only children. The weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathscr{R}^{N\times N}$. For any $i, j \in \mathscr{V}$, $a_{ij} \ge 0$, and $a_{ij} > 0 \Leftrightarrow j \in \mathscr{N}_i$. The Laplacian matrix $L_{\mathscr{G}} = \mathscr{D} - \mathscr{A}$, where $\mathscr{D} = \mathrm{diag}(\sum_{j=1}^{N} a_{1j}, \ldots, \sum_{j=1}^{N} a_{Nj})$.