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Brief paper

Compositional performance certification of interconnected systems using ADMM☆

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ABSTRACT

A compositional performance certification method is presented for interconnected systems using subsystem dissipativity properties and the interconnection structure. A large-scale optimization problem is formulated to search for the most relevant dissipativity properties. The alternating direction method of multipliers (ADMM) is employed to decompose and solve this problem, and is demonstrated on several examples.

1. Introduction

In this paper, compositional analysis is used to certify performance of an interconnection of subsystems as depicted in Fig. 1. The G_i blocks are known subsystems mapping $u_i \mapsto y_i$ and M is a static matrix that characterizes the interconnection topology. The goal of compositional analysis is to establish properties of the interconnected system using only properties of the subsystems and their interconnection. Henceforth, the term "local" is used to refer to properties or analysis of individual subsystems in isolation. Likewise, "global" refers to the entire interconnected system.

Local behavior and global performance are cast and quantified in the framework of dissipative systems (Willems, 1972); specifically the case with quadratic supply rates. The global supply rate is specified by the analyst and dictates the system performance that is to be verified. For example, supply rates can be chosen to characterize L_2-gain, passivity, output-strict passivity, etc., for the input–output pair (d, e). A storage function is then sought to certify dissipativity with respect to the desired supply rate. See Section 2 for definitions of storage functions and supply rates.

A conventional approach to compositional analysis, as presented for example in Anderson, Teixeira, Sandberg, and Papachristodoulou (2011), Dashkovskiy, Rüffer, and Wirth (2007), Sandell, Varaiya, Athans, and Safonov (1978), Vidyasagar (1981) and Willems (1972), is to establish individual supply rates (and storage functions) for which each subsystem is dissipative. Then, a storage function certifying dissipativity of the interconnected system is sought as a combination of the subsystem storage functions.

The method presented here is less conservative because the local supply rates (and storage functions) are optimized with regards to their particular suitability in certifying global properties. Thus, the local certificates are automatically generated, as opposed to being preselected.

Optimizing over the local supply rates (and storage functions) to certify stability of an interconnected system was first introduced in Topcu, Packard, and Murray (2009), with the individual supply rates constrained to be diagonally-scaled induced L_2-norms. This perspective, coupled with dual decomposition, gave rise to a distributed optimization algorithm. We generalize this approach in several ways: certifying dissipativity (rather than stability) of the interconnected system with respect to a quadratic supply rate; searching over arbitrary quadratic supply rates for the local subsystems; and employing ADMM (Boyd, Parikh, Chu, Peleato, & Eckstein, 2011) to decompose and solve the resulting problem.

The ADMM algorithm exposes the distributed certification as a convergent negotiation between parallelizable, local problems

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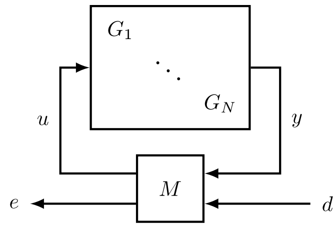

Fig. 1. Interconnected system with input d and output e.

for each subsystem, and a global problem. Each local problem receives a proposed supply rate from the global problem and solves an optimization problem certifying dissipativity of the corresponding subsystem with a supply rate close to the proposed one. The global problem, with knowledge of the interconnection M and the updated supply rates, solves an optimization problem to certify dissipativity of the interconnected system and proposes new supply rates.

In Meissen, Lessard, and Packard (2014) the method presented here was applied to linear systems and ADMM was compared to other distributed optimization methods. In Meissen, Lessard, Arcak, and Packard (2014) this method was extended to nonlinear systems using sum-of-squares (SOS) optimization. Additionally, Meissen, Lessard, Arcak et al. (2014) generalized this approach to systems that are *equilibrium-independent dissipative* (Hines, Arcak, & Packard, 2011).

This paper unifies and expands on the conference papers (Meissen, Lessard, Arcak et al., 2014; Meissen, Lessard, & Packard, 2014). A new theorem shows the proposed method is equivalent to searching for an additively separable storage function for interconnections of linear subsystems. We also demonstrate that the proposed method is tractable and more efficient for large systems than conventional techniques. An extension of the proposed method using integral quadratic constraints is included to allow frequency dependent properties of the subsystems. New examples are presented to demonstrate the results. The convergence properties of ADMM are described and shown to hold for this application.

2. Preliminaries

Dissipative dynamical systems (Willems, 1972). Consider a time-invariant dynamical system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \quad f(0,0) = 0 \\ y(t) &= h(x(t), u(t)), \quad h(0,0) = 0 \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. A *supply rate* is a function $w : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$. A system of the form (1) is *dissipative* with respect to a supply rate w if there exists a differentiable, nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that $V(0) = 0$ and

$$\nabla V(x)^{\mathsf{T}} f(x,u) - w(u, h(x,u)) \le 0 \tag{2}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Eq. (2) is referred to as the *dissipation inequality* and V as a *storage function.*

Equilibrium-independent dissipative (EID) systems (Bürger, Zelazo, & Allgöwer, 2014; Hines et al., 2011). Consider a system of the form

$$\dot{x}(t) = f(x(t), u(t)) \qquad y(t) = h(x(t), u(t)) \tag{3}$$

where there exists a nonempty set $\mathcal{X}^\star \subseteq \mathbb{R}^n$ such that for each $x^\star \in \mathcal{X}^\star$ there exists a unique $u^\star \in \mathbb{R}^m$ such that $f(x^\star, u^\star) = 0$. The *equilibrium state-input map* is then defined as

$$k_u(x) : \mathbb{R}^n \to \mathbb{R}^m \quad \text{such that } u^\star = k_u(x^\star).$$

The system (3) is *EID* with respect to a supply rate w if there exists a nonnegative storage function $V : \mathbb{R}^{2n} \to \mathbb{R}_+$ such that $V(x^\star, x^\star) = 0$ and

$$\nabla_x V(x, x^\star)^{\mathsf{T}} f(x,u) - w(u - u^\star, y - y^\star) \le 0 \tag{4}$$

for all $x^\star \in \mathcal{X}^\star$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$ where $u^\star = k_u(x^\star)$, $y = h(x,u)$, and $y^\star = h(x^\star, u^\star)$.

This definition ensures dissipativity with respect to any possible equilibrium point rather than a particular point. This is advantageous for compositional analysis, since the equilibrium of an interconnection may be hard to compute.

Integral Quadratic Constraints (IQCs) (Megretski & Rantzer, 1997). IQCs are a generalization of the dissipativity framework that capture frequency dependent properties of a system. Let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ be the realization of a stable LTI system Ψ with state η and X be a real symmetric matrix. Then (1) satisfies the IQC defined by $\Pi = \Psi^* X \Psi$ if there exists a nonnegative storage function $V(x,\eta)$ such that $V(0,0) = 0$ and

$$\begin{aligned} &\nabla_x V(x,\eta)^{\mathsf{T}} f(x,u) + \nabla_\eta V(x,\eta)^{\mathsf{T}} \left(\hat{A}\eta + \hat{B} \begin{bmatrix} u \\ y \end{bmatrix} \right) \\ &\le \left(\hat{C}\eta + \hat{D} \begin{bmatrix} u \\ y \end{bmatrix} \right)^{\mathsf{T}} X \left(\hat{C}\eta + \hat{D} \begin{bmatrix} u \\ y \end{bmatrix} \right) \end{aligned} \tag{5}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ where $y = h(x,u)$. In fact, more is true, (5) implies that for all $u \in L_{2e}$, the space of signals that are square integrable on all finite intervals, the signal $z := \Psi \begin{bmatrix} u \\ y \end{bmatrix}$ satisfies $\int_0^T z^{\mathsf{T}} X z dt \ge 0$ for all $T > 0$ with $x(0) = 0$ and $\eta(0) = 0$. Dissipativity is recovered when $\Psi = I_{m+p}$.

SOS programming (Parillo, 2000). For polynomial systems certifying dissipativity can be relaxed to a semidefinite program (SDP) searching for storage functions that are SOS polynomials.

Suppose that f and h in (1) are polynomials. Let $\mathbb{R}[x]$ ($\Sigma[x]$) be the set of polynomials (SOS polynomials) in x. Then certification of dissipativity with respect to a polynomial supply rate, w, can be relaxed to the SOS feasibility program:

$$\begin{aligned} &V(x) \in \Sigma[x] \\ &-\nabla V(x)^{\mathsf{T}} f(x,u) + w(u,y) \in \Sigma[x,u]. \end{aligned} \tag{6}$$

Similarly, as presented in Hines et al. (2011), certifying polynomial systems are EID can be relaxed to:

$$\begin{aligned} &V(x, x^\star) \in \Sigma[x, x^\star] \\ &r(x, u, x^\star, u^\star) \in \mathbb{R}[x, u, x^\star, u^\star] \\ &-\nabla_x V(x, x^\star)^{\mathsf{T}} f(x,u) + w(u - u^\star, y - y^\star) \\ &\quad + r(x, u, x^\star, u^\star) f(x^\star, u^\star) \in \Sigma[x, u, x^\star, u^\star]. \end{aligned} \tag{7}$$

If each state has rational polynomial dynamics,

$$\dot{x}_i = f_i(x,u) = \frac{p_i(x,u)}{q_i(x,u)} \quad \text{for } i = 1, \ldots, n$$

where $p_i \in \mathbb{R}[x,u]$ and $q_i - \epsilon \in \Sigma[x,u]$ for $\epsilon > 0$, then certifying dissipativity of the system with respect to a polynomial supply rate, w, can be relaxed to:

$$\begin{aligned} &V(x) \in \Sigma[x] \\ &-\sum_{i=1}^{n} \nabla_{x_i} V(x) p_i(x,u) \prod_{j \ne i} q_j(x,u) \\ &\quad + \prod_{i=1}^{n} q_i(x,u) w(u,y) \in \Sigma[x,u]. \end{aligned} \tag{8}$$

Similarly to the polynomial case, certifying rational polynomial systems are EID can also be formulated as an SOS feasibility program. Furthermore, certifying a polynomial or rational polynomial system satisfies an IQC can be formulated as a SOS feasibility program.