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Brief paper

Set stability and set stabilization of Boolean control networks based on invariant subsets[☆]

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ABSTRACT

This study addresses the set stability of Boolean networks (BNs) and set stabilization of Boolean control networks (BCNs). Set stability determines whether a BN converges to a given subset, whereas set stabilizability addresses the issue of whether a BCN can be stabilized to a given subset. Many problems can be viewed as special cases of set stability and set stabilization, including synchronization, partial stability, and partial stabilization problems. The concepts of invariant subset and control invariant subset are introduced. Then, algorithms for the largest invariant subset and the largest control invariant subset contained in a given subset are proposed. Based on the invariant subsets obtained, the necessary and sufficient conditions for set stability and set stabilizability are established, and formulae are provided to calculate the shortest transient periods for respective initial states. A design procedure is proposed for finding all the time-optimal set stabilizers. Finally, an example is used to show the application of the proposed results to the synchronization problem of BNs.

1. Introduction

The Boolean Network (BN) was first proposed by Kauffman in order to model genetic regulatory networks (Kauffman, 1969, 1993) and has become a powerful tool in describing, analyzing and simulating cellular networks. The Boolean network finds applications in many fields and has been extensively studied in recent years. See for instance Akutsu, Hayashida, Ching, and Ng (2007), Akutsu, Miyano, and Kuhara (1999), Albert and Barabási (2000), Aldana (2003), Heidel, Maloney, Farrow, and Rogers (2003), Shmulevich, Dougherty, Kim, and Zhang (2002), Shmulevich, Dougherty, and Zhang (2002).

Recently, Cheng presented a new matrix product called the semi-tensor product (STP), which has proven to be very suitable for the analysis and design of BNs (Cheng, 2007; Cheng & Qi, 2010a). Under the framework of STP, any logical function can be equivalently expressed in multi-linear form, allowing a BN to be converted into a discrete-time linear system. The STP has been successfully used to solve many analysis and design problems of Boolean control networks (BCNs) and multi-valued logical systems. See for instance Cheng (2009, 2011), Cheng, Feng, and Lv (2012), Cheng and Qi (2009, 2010b), Cheng, Qi, Li, and Liu (2011), Cheng, Qi, and Li (2011b), Cheng and Zhao (2011), Laschov and Margaliot (2011, 2012), Li and Cheng (2010), Li and Sun (2011), Li, Sun, and Wu (2011), Qi, Cheng, and Hu (2010), Wang, Zhang, and Liu (2012), Xu and Hong (2013), Zhao and Cheng (2013), Zhao, Li, and Cheng (2011) and Zhao, Qi, and Cheng (2010). For a complete introduction to the STP and its applications in different fields, we recommend Cheng, Qi, and Li (2011a) and Cheng, Qi, and Zhao (2012).

The stability and stabilization of BNs are two basic problems that have been investigated in recent years under the framework of STP (Cheng, Qi, Li et al., 2011; Fornasini & Valcher, 2013; Li, Yang, & Chu, 2013, 2014; Qi et al., 2010). In some cases, interest lies in whether a system or a collection of interconnected systems converges to or can be stabilized to a subset of the state space, instead of to a single point. In this study, these are termed set stability and set stabilization, respectively. A typical example of set stability is the synchronization of a collection of locally interconnected systems. Synchronization phenomena are observed in many kinds of complex networks including physical, biological, chemical, technological, and social systems. See for

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example Arenas, Díaz-Guilera, Kurths, Moreno, and Zhou (2008), Garcia-Ojalvo, Elowitz, and Strogatz (2004), Li and Chu (2012), Li, Duan, Chen, and Huang (2010), Morelli and Zanette (2001) and Parriaux, Guillot, and Millérioux (2011) and the references therein. One of the most spectacular biological examples is provided by the synchronized flashing of fireflies observed in nature (Buck, 1938). A range of models has been proposed to model the synchronizing mechanism (Mirollo & Strogatz, 1990; Smith, 1935). Boolean networks have been a focus of attempts to describe collective behavior of this kind in biological networks (Krawitz & Shmulevich, 2009; Teuscher & Capcarrere, 2003).

Until recently, the lack of appropriate mathematical tools made characterizing the synchronization of BNs a challenge. However, the use of STP of matrices (Hong & Xu, 2010) has provided a strictly mathematical treatment for the synchronization problem of BNs for the first time, and many novel results have been reported in the recent literature (Li & Chu, 2012; Li & Lu, 2013; Li, Yang, & Chu, 2012; Xu & Hong, 2013; Zhong, Lu, Huang, & Cao, 2014).

Another problem that frequently arises in systems biology is to determine whether the state variables of a system converge or can be stabilized (Rouche, Habets, Laloy, & Ljapunov, 1977). These two characteristics, termed partial stability and partial stabilization, have been applied to the analysis of consensus in multi-agent systems (Chen, Ge, & Zhang, 2014), and can also be used to characterize the stability of Boolean networks subject to external disturbances, as in Example 3 of this study. Recent developments have been reviewed in Chen and Sun (2014).

This study investigates the set stability and set stabilization of BNs and BCNs, respectively, providing a unified framework for treating a range of problems, including those already mentioned. The main points are as follows:

- The key concepts proposed in this study are the invariant subset for BNs and the control invariant subset for BCNs. It is shown that a BN/BCN is stable/stabilizable with respect to a given subset $\mathcal{M}$ if and only if it is stable/stabilizable with respect to the largest invariant subset/the largest control invariant subset contained in $\mathcal{M}$. Iterative algorithms for the largest invariant subset and the largest control invariant subset contained in a given set are proposed. These form the key steps in solving set stability and stabilizability problems.
- Based on the largest invariant subset algorithms, the necessary and sufficient conditions for set stability and set stabilizability are obtained. One of the advantages of the invariant subset-based results obtained in this study is that they can be used to determine the shortest transient periods for respective initial states.
- A design procedure is proposed to calculate all the time-optimal feedback stabilizers. For any given set stabilizable BCN, a Boolean matrix characterizing all the time-optimal feedback gains is obtained. A logical matrix is a time-optimal feedback gain if and only if it is a logical sub-matrix of the obtained Boolean matrix.

This paper is arranged as follows. In Section 2, some basic concepts and notations are introduced. In Section 3, the problems of set stability and set stabilizability are defined. This section also discusses how the problems of synchronization, node synchronization, and partial stability of BNs are restated as problems of set stability and set stabilization. Section 4 investigates the largest (control) invariant subsets contained in a given subset. In Section 5, criteria for set stability and set stabilizability are discussed. Section 6 proposes a procedure to obtain all the time-optimal set stabilizers. In Section 7, an example of BN synchronization is provided and concluding remarks are made in Section 8.

2. Preliminaries

2.1. Notations and definitions

- $|\mathcal{M}|$ represents the cardinal number of the set $\mathcal{M}$.
- $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ represent the set of positive and nonnegative integers, respectively.
- $\mathrm{Col}_i(A)$ and $\mathrm{Row}_j(A)$ represent the ith column and the jth row of the matrix A, respectively. $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$ represent the set of columns and rows of A, respectively.
- $\mathscr{D} := \{T = 1, F = 0\}$ represents the logical domain.
- $\mathscr{B}_{n\times m}$ represents the set of $n \times m$ Boolean matrices, i.e., all of the matrices $X = (x_{ij})$ with $x_{ij} \in \mathscr{D}$.
- I_n represents the $n \times n$ identity matrix, $\mathbf{1}_n := (1\,1, \ldots\, 1)^T$.
- $\delta_n^i := \mathrm{Col}_i(I_n)$, $\delta_n^0 := 0_{n\times 1}$, $\Delta_n := \mathrm{Col}(I_n)$. In particular, δ_2^1 and δ_2^2 represent the vector forms of the logical values T and F, respectively.
- $\delta_n[j_1, \ldots, j_m]$ represents the Boolean matrix $A \in \mathscr{B}_{n\times m}$ with $\mathrm{Col}_i(A) = \delta_n^{j_i}$.
- $W[m, n]$ represents the swap matrix with index $[m, n]$ defined by $W[m, n] := [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \ldots, I_n \otimes \delta_m^m]$, where $\otimes$ represents the Kronecker product.
- $\mathscr{L}_{n\times m}$ represents the set of $n \times m$ logical matrices, i.e., all of the matrices A with $\mathrm{Col}(A) \subseteq \Delta_n$.
- $A \ltimes B$ represents the semi-tensor product (STP) of matrices A and B defined by $A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p})$, where $n = |\mathrm{Col}(A)|$, $p = |\mathrm{Row}(B)|$, and α is the least common multiple of n and p. When $n = p$, the STP degenerates to the conventional product AB. Thus, the symbol $\ltimes$ may be omitted without causing confusion.
- Assume that $X = (x_{ij}), Y = (y_{ij}) \in \mathscr{B}_{m\times n}$. Then $X \wedge Y := (x_{ij} \wedge y_{ij})$, $X \vee Y := (x_{ij} \vee y_{ij})$. The symbols "$\vee$" and "$\wedge$" represent the logical operators OR and AND, respectively.
- The Boolean addition of Boolean matrices is defined as
$$\begin{cases} \alpha +_{\mathscr{B}} \beta := \alpha \vee \beta, \quad \forall \alpha, \beta \in \mathscr{D} \\ (\mathscr{B}) \displaystyle\sum_{i=1}^{n} \alpha_i := \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n, \quad \forall \alpha_i \in \mathscr{D} \\ X +_{\mathscr{B}} Y = (x_{ij} +_{\mathscr{B}} y_{ij}) \in \mathscr{B}_{m\times n}, \quad \forall X, Y \in \mathscr{B}_{m\times n}. \end{cases}$$
- For any $X \in \mathscr{B}_{m\times n}$, $Y \in \mathscr{B}_{n\times p}$, the Boolean product of X and Y is defined as $X \ltimes_{\mathscr{B}} Y := Z = (z_{ij})_{m\times p} \in \mathscr{B}_{m\times p}$ with $z_{ij} = (\mathscr{B}) \sum_{k=1}^{n} x_{ik} \wedge y_{kj}$.
- For $X \in \mathscr{B}_{n\times n}$, the Boolean powers are defined as
$$X^{(k)} := \underbrace{X \ltimes_{\mathscr{B}} X \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} X}_{k}, \quad \forall k \in \mathbb{Z}_{>0}.$$
Especially, $X^{(0)} := I_n$.
- $\mathrm{Row}_{\Sigma}(M)$ and $\mathrm{Col}_{\Sigma}(M)$ represent the Boolean summations of rows and columns of the Boolean matrix M, respectively.
- For any $\mathbf{F} \in \mathscr{B}_{m\times n}$, a logical matrix $F \in \mathscr{L}_{m\times n}$ is called a logical sub-matrix of $\mathbf{F}$ if $F \wedge \mathbf{F} = F$. Denote by $\mathscr{S}(\mathbf{F})$ the set of all of the logical sub-matrices of $\mathbf{F}$, i.e., $\mathscr{S}(\mathbf{F}) := \{F \in \mathscr{L}_{m\times n} \mid F \wedge \mathbf{F} = F\}$. Especially, for any nonzero $x \in \mathscr{B}_{m\times 1}$, $\mathscr{S}(x) = \{z \in \Delta_m \mid z \wedge x = z\}$. For convenience, define $\mathscr{S}^T(x) := \mathscr{S}(x^T)$ for any $x \in \mathscr{B}_{1\times n}$.

2.2. Algebraic forms of BNs and BCNs

A BN with n nodes is described as
$$\begin{cases} A_1(t+1) = f_1(A_1(t), \ldots, A_n(t)) \\ \vdots \\ A_n(t+1) = f_n(A_1(t), \ldots, A_n(t)) \end{cases} \tag{1}$$