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Linear matrix inequalities for globally monotonic tracking control☆

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ABSTRACT

This paper addresses the problem of achieving monotonic tracking control for any initial condition (also referred to as *global monotonic tracking control*). This property is shown to be equivalent to global non-overshooting as well as to global non-undershooting (i.e., non-overshooting and non-undershooting for any initial condition, respectively). The main objective of this paper is to prove that a stable system is globally monotonic if and only if all the rows of the output matrix are left eigenvectors of the space transition matrix. This property allows one to formulate the design of a controller which ensures global monotonic tracking as a convex optimization problem described by a set of Linear Matrix Inequalities (LMIs).

1. Introduction

Within the context of tracking control, overshoot and undershoot are typically considered to be undesirable features of the response. Even more, in the vast majority of tracking problems, ideally the response monotonically tracks the target. In some situations (e.g. automatic cutting machines, plotters, positioning of a CD disk read/write head to name a few), overshoot and undershoot inevitably lead to an unacceptable performance.

The problem of ensuring that a linear time invariant plant has a non-overshooting and/or a non-undershooting step response has a very long history. Much of the existing literature deals with single input single output (SISO) systems. Papers offering design methods to avoid overshoot or undershoot include (Bement & Jayasuriya, 2004a,b; Darbha, 2003; Darbha & Bhattacharyya, 2002, 2003; Leon de la Barra, 1994). All of these methods assume that the initial state of the system is at rest, and in some cases (Darbha, 2003; Darbha & Bhattacharyya, 2002), the methods avoid overshoot at the cost of greatly slowing the speed of the response, leading to a lengthy settling time. Some seminal results have been recently obtained for non-linear systems using back-stepping techniques in Krstic and Bement (2006).

In contrast to the aforementioned contributions, in Schmid and Ntogramatzidis (2010) a design method is offered for avoiding overshoot for linear systems with multiple inputs and outputs (MIMO), from non-zero initial conditions. In Schmid and Ntogramatzidis (2010) it is shown that for MIMO systems one can achieve arbitrarily fast settling time while also guaranteeing a monotonic response in all components of the output vector for any initial condition. In Schmid and Ntogramatzidis (2012), this technique is extended to the avoidance of undershoot as an additional feature of the control action. In Ntogramatzidis, Trégouët, Schmid, and Ferrante (2014), a geometric approach is used to identify a necessary and sufficient structural condition, solely dependent upon the structure of the system, which ensures global monotonic tracking.

The main result of this paper is to offer a characterization of monotonicity in terms of the left eigenvectors of the state transition matrix. This characterization allows one to formulate the design problem as a computationally tractable necessary and sufficient LMI condition. Interestingly, the form of this LMI condition is compatible with the LMIs of the state feedback multi-objective control framework (Boyd, El, Ghaoui, Feron, & Balakrishnan, 1994; Chilali & Gahinet, 1996), thus enabling to add constraints on the positions of the eigenvalues and/or to bound and optimize some $\mathcal{H}_2$–$\mathcal{H}_\infty$ performance indices.

Even if the continuous-time case is considered throughout the paper, the method presented herein can be straightforwardly adapted to the discrete-time case with only minor differences, that will be pointed out.

2. Problem formulation

Consider the linear time-invariant (LTI) system

$$\Sigma : \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), \quad x(0) = x_0, \\ y(t) = C\,x(t), \end{cases} \tag{1}$$

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where, for all $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output, and A, B and C are appropriate dimensional constant matrices. We will denote with C_i the ith row vector of the output matrix C. This paper is concerned with the problem of monotonically tracking any given constant reference target $r \in \mathbb{R}^p$ from any initial condition $x_0 \in \mathbb{R}^n$. The following standing assumption ensures that the tracking of a constant reference is achievable for any initial condition (He, Chen, & Wu, 2005):

Assumption 2.1. System Σ is right invertible and stabilizable, and Σ has no invariant zeros at the origin.

As shown in Franklin, Powell, and Emami-Naeini (1994), Schmid and Ntogramatzidis (2010), Assumption 2.1 guarantees the existence of $x_{ss} \in \mathbb{R}^n$ and $u_{ss} \in \mathbb{R}^m$ satisfying

$$\begin{cases} 0 = A\, x_{ss} + B\, u_{ss} \\ r = C\, x_{ss} \end{cases} \tag{2}$$

for any step reference $r \in \mathbb{R}^p$. By using the control law[1]

$$u(t) = F\left(x(t) - x_{ss}\right) + u_{ss} \tag{3}$$

and the change of variable $\xi \stackrel{\text{def}}{=} x - x_{ss}$, the following closed-loop autonomous system is obtained:

$$\Sigma_{aut} : \begin{cases} \dot{\xi}(t) = (A + B\,F)\, \xi(t), \quad \xi(0) = \xi_0 \stackrel{\text{def}}{=} x_0 - x_{ss}, \\ y(t) = C\, \xi(t) + r. \end{cases} \tag{4}$$

If $A + B\,F$ is asymptotically stable, x converges to x_{ss}, ξ converges to zero and y converges to r as t goes to infinity. Let the tracking error vector be defined as $\epsilon(t) \stackrel{\text{def}}{=} y(t) - r$, and let us rewrite system (4) as

$$\Sigma_{aut,\epsilon} : \begin{cases} \dot{\xi}(t) = (A + B\,F)\, \xi(t), \quad \xi(0) = \xi_0, \\ \epsilon(t) = C\, \xi(t). \end{cases} \tag{5}$$

We recall that overshoot occurs whenever an output exceeds the target. More precisely, y_k, or the corresponding error component ϵ_k, is said to overshoot if $\epsilon_k(\tilde{t})$ crosses the time axis for some $\tilde{t} \geq 0$, i.e., if and only if there exists $\tilde{t} \geq 0$ such that $\epsilon_k(\tilde{t}) = 0$ and $\dot{\epsilon}_k(\tilde{t}) \neq 0$. We also recall that undershoot means that the output moves further away from the target than its initial distance. Hence, an error component ϵ_k is said to undershoot if and only if there exists $\tilde{t} \geq 0$ such that $\operatorname{sgn} \epsilon_k(\tilde{t}) = \operatorname{sgn} \epsilon_k(0)$ and $|\epsilon_k(\tilde{t})| > |\epsilon_k(0)|$. Finally, ϵ_k is monotonic if and only if $\dot{\epsilon}_k$ never changes sign.

This paper focuses on the design of a state-feedback matrix F for (4) such that, for all initial conditions, (i) $\epsilon(t) \to 0$ for $t \to \infty$; and (ii) $\epsilon_k(t)$ is monotonic for any $\xi_0 \in \mathbb{R}^n$ for all $k \in \{1, \ldots, p\}$. We shall describe this property as *global monotonicity*. In a similar way, we talk about *global non-overshooting* and *global non-undershooting* if non-overshooting and non-undershooting can be achieved in all components of the output for all initial conditions, respectively. It is obvious that monotonicity implies non-overshooting and non-undershooting but not *vice versa*. However, we will prove that, for LTI systems, *global monotonicity*, *global non-overshooting* and *global non-undershooting* are all equivalent concepts.

It is well known that the kth component of the error $\epsilon_k(t)$ in (5) can be written as

$$\epsilon_k(t) = \sum_{i=1}^{\rho} \sum_{j=1}^{m_i} \tilde{\alpha}_{k,i}\, t^{j-1} e^{\lambda_i t} + \sum_{i=1}^{c} \sum_{j=1}^{\tilde{m}_i} \Big[\hat{\alpha}'_{k,i}\, t^{j-1}\, e^{\sigma_i t}\, \cos(\omega_i\, t) + \hat{\alpha}''_{k,i}\, t^{j-1}\, e^{\sigma_i t}\, \sin(\omega_i\, t) \Big], \tag{6}$$

[1] It is easily established that, for every reference r, (3) is equivalent to $u(t) = F\,x(t) + G\,r$ where $G = -\left[C\,(A + B\,F)^{-1} B\right]^{\dagger}$, and where $\dagger$ denotes the Moore–Penrose pseudo inverse.

where $\lambda_1, \ldots, \lambda_\rho$ are the real eigenvalues observable from ϵ_k, with associated algebraic multiplicities $m_1, \ldots, m_\rho$ and where $\mu_1, \ldots, \mu_c, \overline{\mu}_1, \ldots, \overline{\mu}_c$ are the complex eigenvalues observable from ϵ_k and the algebraic multiplicities associated with $\mu_1, \ldots, \mu_c$ are $\tilde{m}_1, \ldots, \tilde{m}_c$, respectively, where $\sigma_i = \mathfrak{Re}\{\mu_i\}$ and $\omega_i = \mathfrak{Im}\{\mu_i\}$. It is easily established that $\tilde{\alpha}_{k,i}$, $\hat{\alpha}'_{k,i}$, $\hat{\alpha}''_{k,i}$ are functions of the initial conditions.

Remark 2.1. As proved e.g. in Ntogramatzidis et al. (2014), the real coefficients $\tilde{\alpha}_{k,i}$, $\hat{\alpha}'_{k,i}$ and $\hat{\alpha}''_{k,i}$ in (6) can be made arbitrary by choosing suitable initial conditions. Therefore, if the mode $t^j e^{\lambda_i t}$ appears in ϵ_k for a certain $j \geq 1$, for some initial conditions all the modes $t^l e^{\lambda_i t}$ with $0 \leq l \leq j - 1$ will also appear in ϵ_k. Similarly, if the mode $t^{j-1}\, e^{\sigma_i t}\, \cos(\omega_i\, t)$ appears in ϵ_k, for some initial condition the mode $t^{j-1}\, e^{\sigma_i t}\, \sin(\omega_i\, t)$ will appear in ϵ_k as well, and *vice versa*.

If the kth component of the response is monotonic from any initial condition, only components of the form $e^{\lambda_i t}$ for real λ_i can appear in each ϵ_k. Indeed, for any real $\lambda_i < 0$, the function $t^{j-1}\, e^{\lambda_i t}$ is monotonic only if $j = 1$, and the functions $t^{j-1} e^{\sigma_i t} \cos(\omega_i\, t)$ and $t^{j-1} e^{\sigma_i t} \sin(\omega_i\, t)$ are monotonic only if $j = 1$ and $\omega_i = 0$. Thus, the remaining case is $\epsilon_k(t) = \sum_{i=1}^{\rho} \tilde{\alpha}_{k,i} e^{\lambda_i t}$, where the real coefficients $\tilde{\alpha}_{k,i}$ can be made arbitrary by choosing suitable initial conditions. From Lemma A.1 of Schmid and Ntogramatzidis (2010), it follows that if $\epsilon_k(t)$ is a linear combination of two or more negative real exponential functions, it will change sign (and hence not be monotonic) for some coefficients $\tilde{\alpha}_{k,i}$. Thus, for each $k \in \{1, \ldots, p\}$ we must have $\epsilon_k(t) = \tilde{\alpha}_{k,i} e^{\lambda_i t}$ for some eigenvalue λ_i and some real coefficient $\alpha_k \stackrel{\text{def}}{=} \tilde{\alpha}_{k,i}$. In conclusion, global monotonicity can be obtained if and only if

$$\epsilon(t) = \begin{bmatrix} \alpha_1\, e^{\lambda_1 t} \\ \vdots \\ \alpha_p\, e^{\lambda_p t} \end{bmatrix} \tag{7}$$

where $\lambda_1, \ldots, \lambda_p$ are real and negative (up to a re-indexing of the closed-loop eigenvalues). Accordingly, we can define the problem tackled in this paper as follows.

Problem 2.1. Let $\lambda_1, \ldots, \lambda_p \in \mathbb{R}^-$. Find a matrix F such that applying (3) to Σ yields an asymptotically stable closed-loop system $\Sigma_{aut,\epsilon}$ for which, from all initial conditions and for all step references, the tracking error is as in (7).

Interestingly, the next result shows that – as already anticipated – global monotonicity, global non-overshooting and global non-undershooting are all equivalent concepts.

Theorem 2.1. *Let u be given by (3). The following statements are equivalent:*

(1) *Σ_{aut} is globally monotonic;*
(2) *Σ_{aut} is globally non-overshooting;*
(3) *Σ_{aut} is globally non-undershooting.*

Proof. Since a monotonic response is non-overshooting and non-undershooting, it is obvious that (1) implies (2) and (3). Let us prove that (2) or (3) implies (1). We show in particular that if Σ_{aut} is not globally monotonic, it cannot be globally non-overshooting nor non-undershooting. If Σ_{aut} is not globally monotonic, at least one component of the tracking error ϵ_k contains (i) a mode $t^h\, e^{\lambda t}$ for some $h \in \mathbb{N}$; or (ii) a mode $t^h\, e^{\sigma_i t}\, \cos(\omega_i\, t)$ for some $h \in \mathbb{N}$; (iii) a mode $t^h\, e^{\sigma_i t}\, \sin(\omega_i\, t)$ for some $h \in \mathbb{N}$; or (iv) the sum of two or more modes. In view of Remark 2.1, we can choose an arbitrary one of the modes appearing in (6) and select the initial condition in such a way that $\epsilon_k(t)$ coincides exactly with that mode. Thus, it suffices to show that the error components $\epsilon_k(t) = \alpha\, t^h\, e^{\lambda t}$, $\epsilon_k(t) = \alpha\, t^h\, e^{\sigma_i t}\, \cos(\omega_i\, t)$, $\epsilon_k(t) = \alpha\, t^h\, e^{\sigma_i t}\, \sin(\omega_i\, t)$ and $\epsilon_k(t) =$