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Stability of stochastic nonlinear systems in cascade with not necessarily unbounded decay rates☆

Hiroshi Ito[a,1], Yuki Nishimura[b]

[a] Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, 820-8502, Japan

[b] Department of Mechanical Engineering, Kagoshima University, 1-21-40 Korimoto, Kagoshima, 890-0065, Japan

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ABSTRACT

This paper develops tools to verify stability and robustness of cascaded nonlinear stochastic systems based on Lyapunov functions. Constituent systems are formulated in terms of integral input-to-state stability (iISS) and input-to-state stability (ISS) which are popular notions for both stochastic and deterministic systems. This paper highlights differences between the stochastic and the deterministic cases. In contrast to deterministic systems, it is demonstrated that assuming ISS systems having unbounded decay rates in dissipation inequalities is restrictive. Taking this fact into account, stability criteria are formulated without assuming unboundedness of decay rates, so that ISS systems with bounded decay rates and iISS systems which are not ISS are covered in a unified manner. Stability criteria for stochastic cascades involve the growth rate conditions at connecting channels. This paper clarifies how noise diffusion fields affect the growth rate conditions and the influence depends on definition of stochastic robustness.

1. Introduction

Stochastic differential equations of Itô form are useful for modeling fluctuation and uncertainty arising in dynamical systems. Control systems are integration of dynamical modules. Combining properties of modules together is typically the most efficient way to understand and synthesize large systems. Therefore, it is undoubtedly important to study the influence of fluctuation and uncertainty on system interconnection in the framework of stochastic differential equations. Notions of integral input-to-state stability (iISS) and input-to-state stability (ISS)[2] provide one of popular frameworks for studying stability and robustness of interconnections of deterministic systems. Feedback interconnections were tackled in such a framework for stochastic nonlinear systems (Ito & Nishimura, in press-a; Wu, Karimi, & Shi, 2013; Wu, Xie, & Zhang, 2007; Yu & Xie, 2010; Yu, Xie, & Duan, 2010). Importantly, for deterministic systems, it is known that compared with feedback interconnections, stability and robustness of cascaded systems can be established under milder conditions (Arcak, Angeli, & Sontag, 2002; Chaillet & Angelli, 2008; Ito, 2010; Panteley & Loría, 1998, 2001; Sontag & Teel, 1995). For instance, a cascade connection of ISS systems is always ISS. A cascade of an iISS driven system and an ISS driving system is iISS whenever the connecting channel fulfills a growth rate condition. One of the aims of this paper is to investigate whether there are similar facts for stochastic systems. This paper also aims at demonstrating fundamental distinctions from deterministic cases.

Lyapunov-type methods have been extensively studied in the literature of control of stochastic nonlinear systems, e.g. Ferreira, Arcak, and Sontag (2012), Khasminskii (2012), Krstić and Deng (1998), Liu, Zhang, and Jiang (2008), Spiliotis and Tsinias (2003), Tang and Basar (2001), Tsinias (1998), Wu et al. (2007), Xie and Tian (2009), Yu and Xie (2010) and Yu et al. (2010) to name a few. It is widely known that replacing the derivative of a Lyapunov function along trajectories of a deterministic system by an infinitesimal generator involving a Hessian term is the technical key to dealing with stochastic systems. In particular, a stability criterion for cascaded systems was proposed in Liu et al. (2008) when subsystems are ISS in probability (Tang & Basar, 2001). Although cascaded systems had no feedback loop, the criterion was referred to a small-gain condition. This terminology might not be intuitive, but the result nicely

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E-mail addresses: hiroshi@ces.kyutech.ac.jp (H. Ito), yunishi@mech.kagoshima-u.ac.jp (Y. Nishimura).

1 Tel.: +81 948 29 7717; fax: +81 948 7709.

2 iISS generalizes ISS in the sense that a system is ISS only if it is iISS (Sontag, 1998).

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described how the Hessian term bothers one in dealing with interconnected systems. Its idea of coping with the Hessian sharply contrasts to the approach employed in Wu et al. (2013) assuming concavity of system gains to get rid of the Hessian term.

It is known that deterministic ISS systems always admit radially unbounded decay rates in their dissipation inequalities for appropriately chosen Lyapunov functions (Sontag & Wang, 1995). Based on this fact, following the seminal work for deterministic systems (Sontag & Teel, 1995), cascades of stochastic ISS systems are formulated in Liu et al. (2008) with radially unbounded decay rates. However, for a stochastic cascade, it has not been known whether it is reasonable to assume the unboundedness of decay rates. This paper gives a characterization demonstrating that the unboundedness is demanding for stochastic systems even if ISS is assumed.

Bounded decay rates have sometimes been addressed in preceding studies such as Yu and Xie (2010) and Yu et al. (2010). However, those studies cope with the Hessian only if decay rates are unbounded. Indeed, the typical idea to tackle bounded decay rates or non ISS systems is as simple as checking if summing up (i.e., linear combination) Lyapunov functions of individual subsystems establishes stability of interconnected systems. The linear restriction merely renders the coefficient of the troublesome Hessian term identically zero. For deterministic systems, it is known that the effectiveness of linear combination is very limited, and the linear combination results in stability criteria which are far more conservative than those utilizing nonlinear combination (Ito, 2006; Ito & Jiang, 2009; Jiang, Mareels, & Wang, 1996; Praly, Carnevale, & Astolfi, 2010). In fact, no linear combination can explain the fact that cascade of ISS is always ISS. No linear combination can explain the aforementioned fact on an iISS system driven by an ISS system either. This paper provides a way to effectively use nonlinear combination of Lyapunov functions for stochastic systems to obtain less conservative criteria for cascades. This is done in an iISS framework for stochastic systems where decay rates are allowed to be merely positive definite. Preliminary results of the material in this paper were reported in Ito and Nishimura (2014) without any proofs. In addition to providing proofs and improving examples, this paper strengthens some results and extends iISS/ISS to practical iISS/ISS which not only admit biases, but also allow diffusion fields of stochastic noises to be non-vanishing at the origin. Proofs are presented in appendices. Some of them are omitted due to space limitation.

Notation: Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For a given vector x of the n-dimensional real vector space $\mathbb{R}^n$, the Euclidean norm is denoted by $|x|$. For a matrix X, $|X|_{\mathcal{F}}$ denotes the Frobenius norm defined by $|X|_{\mathcal{F}} = \sqrt{\mathrm{Tr}\{X^T X\}}$, where the superscript T indicates the transpose of a matrix, and Tr is the trace of a square matrix. The symbol **Id** denotes the identity function on $\mathbb{R}_+$. A continuous function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{P}$ and one writes $\zeta \in \mathcal{P}$ if $\zeta(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$, and $\zeta(0) = 0$. A continuous function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing. It is of class $\mathcal{K}_\infty$ if, in addition, $\lim_{s\to\infty} \zeta(s) = \infty$. A continuous function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\eta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, $\eta(s, \cdot)$ is decreasing and $\lim_{t\to\infty} \eta(s, t) = 0$. For any given $\zeta \in \mathcal{K}$, define the operator $\zeta^{\ominus}$: $[0, \infty] \to [0, \infty]$ as $\zeta^{\ominus}(s) = \sup\{v \in [0, \infty) : s \geq \zeta(v)\}$. By definition, one has $\zeta^{\ominus}(s) = \zeta^{-1}(s)$ for $s < \lim_{\tau\to\infty} \zeta(\tau)$, and $\zeta^{\ominus}(s) = \infty$ elsewhere. Any non-decreasing continuous function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is extended to the operator ζ: $[0, \infty] \to [0, \infty]$ as $\zeta(s) = \sup_{v\in\{w\in[0,\infty):w\leq s\}} \zeta(v)$.

2. Definitions

2.1. Robustness w.r.t. deterministic disturbance

Consider the stochastic differential equation of Itô form

$$dx = f(x, r)dt + h(x)dw, \tag{1}$$

where $x(t) \in \mathbb{R}^N$ is the state, and $r(t) \in \mathbb{R}^M$ is the deterministic disturbance which is any measurable, locally essentially bounded function $r : \mathbb{R}_+ \to \mathbb{R}^M$. The drift field $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ and the diffusion field $h : \mathbb{R}^N \to \mathbb{R}^{N\times S}$ are locally Lipschitz. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a sample space Ω, a probability measure $\mathbb{P}$, a σ-algebra $\mathcal{F}$, and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t) = [w_1(t), w_2(t), \ldots, w_S(t)]^T$ be an S-dimensional standard Brownian motion defined on the probability space. This paper uses the following definition (Liu et al., 2008; Tang & Basar, 2001).

Definition 1. System (1) is said to be input-to-state practically stable (ISpS) in probability if for each $\epsilon \in (0, 1)$, there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and a real number $b \geq 0$ such that

$$\mathbb{P}\left\{|x(t)| < \beta(|x(0)|, t) + \gamma\left(\sup_{\tau\in[0,t]} |r(\tau)|\right) + b\right\} \geq 1 - \epsilon, \quad \forall t \in \mathbb{R}_+,\ x(0) \in \mathbb{R}^N \setminus \{0\}. \tag{2}$$

If $b = 0$, System (1) is said to be input-to-state stable (ISS) in probability.

In this paper, system (1) is said to be 0-GAS in probability if (2) is satisfied for $r(t) \equiv 0$ and $b = 0$ (see Krstić & Deng, 1998).

Definition 2. System (1) is said to be integral-input-to-state practically stable (iISpS) in probability if for each $\epsilon \in (0, 1)$, there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function μ, a class $\mathcal{K}_\infty$ function χ and a real number $b \geq 0$ such that

$$\mathbb{P}\left\{\chi(|x(t)|) < \beta(|x(0)|, t) + \int_0^t \mu(|r(\tau)|)d\tau + bt\right\} \geq 1 - \epsilon, \quad \forall t \in \mathbb{R}_+,\ x(0) \in \mathbb{R}^N \setminus \{0\}. \tag{3}$$

If $b = 0$, System (1) is said to be integral input-to-state stable (iISS) in probability.

The above property with $b = 0$ is an exact analog of iISS for deterministic systems (Sontag, 1998). Since stochastic noise whose magnitude can be arbitrarily large instantaneously may not allow the initial state to fade out, the following variant is useful.

Definition 3. System (1) is said to be quasi-integral-input-to-state practically stable (quasi-iISpS) in probability if there exists a constant $R > 0$ satisfying the following: for each $\epsilon \in (0, 1)$, there exist a class $\mathcal{KL}$ function β, class $\mathcal{K}$ functions $\overline{\beta}, \mu, \gamma$, a class $\mathcal{K}_\infty$ function χ and a real number $b \geq 0$ such that

$$\mathbb{P}\left\{\chi(|x(t)|) < \overline{\beta}(|x(0)|) + \int_0^t \mu(|r(\tau)|)d\tau + bt\right\} \geq 1 - \epsilon, \quad \forall t \in \mathbb{R}_+,\ x(0) \in \mathbb{R}^N \setminus \{0\} \tag{4}$$

$$\|r\| < R \ \Rightarrow\ (2). \tag{5}$$

If $b = 0$, System (1) is said to be quasi-integral input-to-state stable (quasi-iISS) in probability.

Here, $\|\cdot\|$ denotes the (essential) supremum norm. Property (4) does not guarantee 0-GAS in probability even if $b = 0$. It is stressed that the functions β, $\overline{\beta}$, γ, μ, χ and constant b in (2)–(4) may depend on ϵ. Usually, it is inevitable that the smaller ϵ is, the larger β, $\overline{\beta}$, γ, μ and b should become. The bias $b \geq 0$ in all definitions in this subsection allows the diffusion field h to be non-vanishing at the origin, In fact, $b = 0$ implies $h(0) = 0$ in all definitions.