Automatica 62 (2015) 11–17

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# Convergence of max–min consensus algorithms[☆]

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ARTICLE INFO

Article history:
Received 4 February 2014
Received in revised form
25 August 2015
Accepted 30 August 2015

Keywords:
Consensus algorithms
Time-dependent graphs
Asymptotic convergence

ABSTRACT

In this paper, we propose a distributed max–min consensus algorithm for a discrete-time $n$-node system. Each node iteratively updates its state to a weighted average of its own state together with the minimum and maximum states of its neighbors. In order for carrying out this update, each node needs to know the positive direction of the state axis, as some additional information besides the relative states from the neighbors. Various necessary and/or sufficient conditions are established for the proposed max–min consensus algorithm under time-varying interaction graphs. These convergence conditions do not rely on the assumption on the positive lower bound of the arc weights.



## 1. Introduction

In the past decade, distributed consensus algorithms have been extensively studied in the literature, due to its wide applicability in engineering, computer science, and social science (DeGroot, 1974; Diekmann, Frommer, & Monien, 1999; Golub & Jackson, 2007; Jadbabaie, Lin, & Morse, 2003; Tsitsiklis, Bertsekas, & Athans, 1986). In many cases consensus algorithms seek to compute the average of the nodes' initials over the network (Jadbabaie et al., 2003; Tsitsiklis et al., 1986), and various efforts have been devoted to analyzing how the underlying communication graphs influence the convergence or the convergence rate for both continuous-time and discrete-time agent dynamics (Cao, Morse, & Anderson, 2008a,b; Morse, 2005; Nedic, Olshevsky, Ozdaglar, & Tsitsiklis, 2009; Olfati-Saber & Murray, 2004; Ren & Beard, 2005). Weighted average consensus algorithms, also draw attentions in which all nodes eventually reach an agreement as a weighted average of the initial values (Ren & Beard, 2005). Weighted average consensus is resulted from the missing of balance in the communication graph (Ren & Beard, 2005), and it has been shown that even a weighted agreement still leads to certain wisdom for the networks under quite general conditions (Golub & Jackson, 2007).

A great advantage in distributed consensus algorithms is that they do not rely on a centralized coordinate system. Each node can carry on the computation using only relative state information from its neighbors. A convenient way of modeling the switching node interactions is to assume that the communication graphs are defined by a sequence of time-dependent graphs over the node set. The connectivity of this sequence of graphs plays an important role for the network to reach consensus. Joint connectivity, i.e., connectivity of the union graph over time intervals, has been considered, and various convergence conditions have been established (Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005; Cao et al., 2008a,b; Jadbabaie et al., 2003; Morse, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Tsitsiklis et al., 1986).

On the other hand, it is however true that in most existing works, the convergence of consensus algorithms highly depends on some critical conditions on network information flow. Most asymptotic convergence results are based on the assumption that the arc weights always have a positive lower bound over time, and particularly it is commonly assumed that the underlying communication graphs always keep self-loops reflected as node self-confidence in the node state updates (Blondel et al., 2005; Cao et al., 2008a,b; Jadbabaie et al., 2003; Ren & Beard, 2005; Tsitsiklis et al., 1986).

In this paper, we propose a distributed max–min consensus algorithm for an $n$-node system. In the proposed algorithm, each node iteratively updates its state to a weighted average of its

☆ This work has been supported in part by the Knut and Alice Wallenberg Foundation, the Swedish Research Council, and KTH SRA TNG. The material in this paper was partially presented at the 2013 American Control Conference, June 17–19, 2013, Washington, DC, USA (Shi & Johansson, 2013a,b). This paper was recommended for publication in revised form by Associate Editor Claudio de Persis under the direction of Editor Christos G. Cassandras.

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http://dx.doi.org/10.1016/j.automatica.2015.09.012
0005-1098/

own state together with the minimum and maximum states of its neighbors. This dynamics provides a natural model for extreme-biased opinion evolution over social networks. In classical DeGroot's model (DeGroot, 1974), weights of exchanged opinions are put on different nodes during interactions, without identifying specific opinions. Variants to DeGroot's belief evolution taking into account biases in different opinions have been considered. Krause's model (Krause, 1997) introduced state-dependent interactions where nodes interact with neighbors within certain range of opinions and therefore put a zero weight to opinions outside this interaction range. Recent work (Dandekar, Goel, & Lee, 2013) proposes a biased social interaction model with greater interactions between like-minded individuals and shows that this biased model often leads to polarization of opinions. The proposed max–min consensus algorithm actually defines extreme-biased belief evolution in which nodes put weights only on the extreme (max and min) opinions in the neighborhood, right opposite to the homophily effects studied in Dandekar et al. (2013) and Krause (1997). We show that this extreme-biased dynamics leads to convergence to an agreement under more general conditions, compared to DeGroot type updates.

Compared to standard consensus algorithms, in the proposed algorithm each node needs to know the positive direction of the state axis, as some additional information besides the relative states from the neighbors. This piece of additional information is indeed centralized, but obviously it is not expensive in many practical applications. Various necessary and/or sufficient conditions are established for the proposed max–min consensus algorithm under time-dependent interaction graphs. These conditions are consistent with the infinite flow property and persistent connectivity conditions in the literature which are utilized to study consensus algorithms (Hendrickx & Tsitsiklis, 2013; Martin & Girard, 2013; Touri & Nedic, 2011, 2012). The derived convergence conditions for directed graphs do not rely on the condition on the positive lower bound of the arc weights, which usually show up for the study of standard consensus algorithms. In other words, this small amount of centralized information has brought nontrivial relaxation to the convergence requirements, which is consistent with the recent study on the role of centralized information in queueing systems (Tsitsiklis & Xu, 2011).

The rest of the paper is organized as follows. In Section 2 we introduce the considered network model and the proposed max–min consensus algorithm. Some impossibilities of finite-time or asymptotic consensus are established in Section 3. Then sufficient convergence conditions for asymptotic consensus are established for time-dependent graphs in Section 4.[2] Finally some concluding remarks are given in Section 5.

## 2. Problem definition

In this section, we introduce the network model, the considered algorithm, and define the problem of interest.

### 2.1. Network

We first recall some concepts and notation in graph theory (Godsil & Royle, 2001). A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An element $e = (i, j) \in \mathcal{E}$ is called an *arc* from node $i \in \mathcal{V}$ to $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ of nodes $v_i \in \mathcal{V}$ and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, k$, the sequence is called a (directed) *path* of *length* $k$. If there exists a path from node $i$ to node $j$, then node $j$ is said to be reachable from node $i$. Each node is thought to be reachable by itself. A node $v$ from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. A digraph $\mathcal{G}$ is said to be *strongly connected* if node $i$ is reachable from $j$ for any two nodes $i, j \in \mathcal{V}$; *quasi-strongly connected* if $\mathcal{G}$ has a center (Berge & Ghouila-Houri, 1965). The *distance* from $i$ to $j$ in a digraph $\mathcal{G}$, $d(i, j)$, is the length of a shortest simple path from $i$ to $j$ if $j$ is reachable from $i$, and the *diameter* of $\mathcal{G}$ is $\mathrm{diam}(\mathcal{G}) = \max\{d(i, j) | i, j \in \mathcal{V},\ j \text{ is reachable from } i\}$. The union of two digraphs with the same node set $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$. A digraph $\mathcal{G}$ is said to be bidirectional if for every two nodes $i$ and $j$, $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. A bidirectional graph $\mathcal{G}$ is said to be *connected* if there is a path between any two nodes. A bidirectional underlying graph of a directed graph $\mathcal{G}$ is obtained by replacing all directed edges of $\mathcal{G}$ with bidirectional edges.

Consider a network with node set $\mathcal{V} = \{1, 2, \dots, n\}$, $n \geq 3$. Time is slotted. Denote the state of node $i$ at time $k \geq 0$ as $x_i(k) \in \mathbb{R}$. Then $x(k) = \big(x_1(k) \dots x_n(k)\big)^T$ represents the network state. The interactions among the nodes are determined by a given sequence of digraphs with node set $\mathcal{V}$, denoted as $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$, $k = 0, 1, \dots$.

Throughout this paper, we call node $j$ a *neighbor* of node $i$ if there is an arc from $j$ to $i$ in the graph. Each node is supposed to always be a neighbor of itself. Let $\mathcal{N}_i(k)$ represent the neighbor set of node $i$ at time $k$.

### 2.2. Algorithm

In this paper, we propose the following max–min consensus algorithm for node $i$'s update:

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + \big(1 - \eta_k - \alpha_k\big) \max_{j \in \mathcal{N}_i(k)} x_j(k), \tag{1}$$

where $\alpha_k, \eta_k \geq 0$ and $\alpha_k + \eta_k \leq 1$. We denote the set of all algorithms of the form (1) by $\mathcal{A}$, when the parameters $(\alpha_k, \eta_k)$ take values as $\eta_k \in [0, 1]$, $\alpha_k \in [0, 1 - \eta_k]$. We use $\mathcal{A}_1$ to denote the set of algorithms in the form of (1) with parameters $\eta_k \in (0, 1]$, $\alpha_k \in [0, 1 - \eta_k]$ and use $\mathcal{A}_2$ to denote the set of algorithms in the form of (1) with parameters $\eta_k = 0$ for $k \geq 0$ and $\alpha_k \in [0, 1]$.

Algorithm (1) provides a natural model for extreme-biased opinion dynamics in social networks, where the biased node only assigns weights to extreme opinions in its neighborhood.

### 2.3. Problem

Let $\big\{x(k; x^0) = \big(x_1(k; x^0) \dots x_n(k; x^0)\big)^T\big\}_{k_0}^{\infty}$ be the sequence generated by (1) for initial time $k_0$ and initial value $x^0 = x(k_0) = \big(x_1(k_0) \dots x_n(k_0)\big)^T \in \mathbb{R}^n$. We will identify $x(k; x^0)$ as $x(k)$ in the following discussions. We introduce the following definition on the convergence of the considered algorithm.

**Definition 1.** (i) Asymptotic consensus is achieved for Algorithm (1) for initial condition $x(k_0) = x^0 \in \mathbb{R}^n$ if there exists $z_*(x^0) \in \mathbb{R}$ such that $\lim_{k \to \infty} x_i(k) = z_*$, $i = 1, \dots, n$. Global asymptotic consensus is achieved if asymptotic consensus is achieved for all $k_0 \geq 0$ and $x^0 \in \mathbb{R}^n$.

(ii) Finite-time consensus is achieved for Algorithm (1) for initial condition $x(k_0) = x^0 \in \mathbb{R}^n$ if there exist $z_*(x^0) \in \mathbb{R}$ and an

[2] The proposed max–min consensus algorithms can also be studied for a type of state-dependent graphs as shown in Shi and Johansson (2013b).