Automatica 62 (2015) 304–311

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Integral fractional pseudospectral methods for solving fractional optimal control problems☆

Xiaojun Tang [a,1], Zhenbao Liu [a], Xin Wang [b]

[a] School of Aeronautics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China

[b] School of Astronautics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China

ARTICLE INFO

Article history:
Received 7 November 2014
Received in revised form
21 May 2015
Accepted 22 August 2015
Available online 11 November 2015

Keywords:
Optimal control
Pseudospectral methods
Pseudospectral integration matrices

ABSTRACT

The main purpose of this work is to provide a unified framework and develop integral fractional pseudospectral methods for solving fractional optimal control problems. As a generalization of conventional pseudospectral integration matrices, fractional pseudospectral integration matrices (FPIMs) and their efficient and stable computation are the key to our new approach. In order to achieve this goal, we take a special and smart way to compute FPIMs. The essential idea is to transform the fractional integral of Lagrange interpolating polynomials through a change of variables into their Jacobi-weighted integral which can be calculated exactly using the Jacobi–Gauss quadrature. This, together with the stable barycentric representation of Lagrange interpolating polynomials and the explicit barycentric weights for the Gauss-, flipped Radau-, and Radau-type points corresponding to the Jacobi polynomials, leads to an exact, efficient, and stable scheme to compute FPIMs even at millions of Jacobi-type points. Numerical results on two benchmark optimal control problems demonstrate the performance of the proposed pseudospectral methods.

1. Introduction

Optimal control problems arise naturally in various areas of science, engineering, and mathematics. Considerable work has been done in the area of integer optimal control problems (IOCPs) whose dynamics are described by conventional integer differential equations (see, e.g., Bryson & Ho, 1975; Lewis, Vrabie, & Syrmos, 2012 and the references therein). Recently, it has been demonstrated that fractional differential equations are more accurate than integer differential equations to describe the dynamic behavior of many real-world physical systems (Tarasov, 2011). As defined in Agrawal (2004), fractional optimal control problems (FOCPs) are a subclass of optimal control problems whose dynamics are described by fractional differential equations. There are various definitions of fractional derivatives and the two most important types are the Riemann–Liouville derivatives and the Caputo derivatives. It is important to point out that, unlike the integer derivatives which are locally defined on the epsilon neighborhood of a chosen point, the fractional derivatives are globally defined by a definite integral over the whole domain. More background information on the fractional calculus can be found in Sabatier, Agrawal, and Tenreiro Machado (2007).

It is well known that the analytical solution of FOCPs generally does not exist except for special cases, and therefore, numerical methods to obtain an approximate solution have become the preferred approach for solving FOCPs. In general, numerical methods for solving FOCPs fall into two major categories: indirect methods and direct methods. In an indirect method, necessary optimality conditions of a FOCP are derived by using the fractional calculus of variations, leading to a fractional multiple-point boundary value problem that is then solved to obtain candidate optimal solutions. In a direct method, a continuous FOCP is transcribed to a finite-dimensional nonlinear programming problem (NLP) through the parameterization of the state and/or control variables in some manner, and the resulting NLP is then solved using well-known optimization software. Inspired by the aforementioned *global* property of the fractional derivatives, it is quite natural for us to conjure up that global direct methods, such as pseudospectral methods, are perhaps more suitable than other

☆ This work was partly supported by the Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2014JQ8366) and the National Natural Science Foundation of China (Grant No. 61473228). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Editor Berç Rüstem.

E-mail addresses: nputxj@nwpu.edu.cn (X. Tang), liuzhenbao@nwpu.edu.cn (Z. Liu), wangxinfengdz@126.com (X. Wang).

[1] Tel.: +86 13571893786; fax: +86 29 88492344.

http://dx.doi.org/10.1016/j.automatica.2015.09.007
0005-1098/

methods for solving FOCPs. Yet unfortunately, to the extent of our knowledge, pseudospectral methods for solving FOCPs have not received attention although they have been extensively used in the numerical solution of IOCPs such as the Gauss pseudospectral method (GPM) (Benson, Huntington, Thorvaldsen, & Rao, 2006) and the Radau pseudospectral method (RPM) (Garg et al., 2011), to name a few but two. There are two primary implementation forms for pseudospectral methods: differential and integral. In this work, we focus on the latter, i.e., integral pseudospectral methods.

The motivation of this paper is to provide a unified framework and develop integral fractional pseudospectral methods for solving FOCPs. As a generalization of conventional pseudospectral integration matrices (Francolin, Benson, Hager, & Rao, 2015; Garg et al., 2010; Tang, 2015), fractional pseudospectral integration matrices (FPIMs) and their efficient and stable computation are the key to our new approach. We point out that Garg et al. (2010) touched on the framework for solving IOCPs using pseudospectral methods in a very different context of the equivalence between differential and integral pseudospectral methods as well as their key mathematical properties. Specifically, the main contributions of this work are as follows:

(i) We provide a unified framework for solving FOCPs using integral pseudospectral methods, filling the gap between the numerical solution of FOCPs and IOCPs. This is actually feasible because IOCPs are recovered from FOCPs when fractional order $\gamma = 1$. Under the framework, we develop novel integral fractional pseudospectral methods using collocation at the Gauss- and flipped Radau-type points, respectively. The latest integral GPM (IGPM) and integral flipped RPM (IFRPM) presented in Francolin et al. (2015) can be viewed as special cases of the corresponding integral fractional pseudospectral methods with $\gamma = 1$. Moreover, this unified framework even allows for solving hybrid optimal control problems (HOCPs) whose dynamics contain both the fractional and integer derivatives (Pooseh, Almeida, & Torres, 2014), and also inspires us to extend the framework of Garg et al. (2010) to the case of FOCPs in the future.

(ii) We propose the notion of FPIMs and take a special and smart way to compute FPIMs efficiently and stably. The essential idea is to transform the fractional integral of Lagrange interpolating polynomials through a change of variables into their Jacobi-weighted integral which is then calculated exactly using the Jacobi–Gauss (JG) quadrature. As a result, the computation of FPIMs is reduced to the calculation of Lagrange interpolating polynomials which can be further represented in the stable barycentric form. In particular, the barycentric weights for the Gauss-, flipped Radau-, and Radau-type points can be expressed explicitly in terms of the corresponding quadrature weights for classic orthogonal polynomials (Wang, Huybrechs, & Vandewalle, 2014) such as the Jacobi polynomials. Therefore, this novel approach leads to an exact, efficient, and stable scheme to compute FPIMs even at millions of Jacobi-type points.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented for subsequent developments. In Section 3, the differential and integral forms of scaled FOCP are described. The definitions and computation of FPIMs are presented in Section 4. In Section 5, the implementation details of integral fractional pseudospectral methods are described. Numerical results on two benchmark problems are shown in Section 6. Finally, Section 7 contains some concluding remarks.

2. Some preliminaries

In this work, the left and right Riemann–Liouville fractional integrals of real order $\gamma \geq 0$ of a function $h(t), t \in [t_0, t_f]$ are denoted by ${}_{t_0}\mathcal{I}_t^{\gamma} h(t)$ and ${}_t\mathcal{I}_{t_f}^{\gamma} h(t)$, respectively. Accordingly, the left and right Caputo fractional derivatives of real order $\gamma \in (n-1, n], n = \lceil \gamma \rceil \in \mathbb{N}$ of a function $h(t)$ are denoted, respectively, by ${}_{t_0}^{C}\mathcal{D}_t^{\gamma} h(t)$ and ${}_t^{C}\mathcal{D}_{t_f}^{\gamma} h(t)$ where $\lceil \gamma \rceil$ denotes the smallest integer greater than or equal to γ. In particular, we have ${}_{t_0}\mathcal{I}_t^{0} h(t) = {}_t\mathcal{I}_{t_f}^{0} h(t) = {}_{t_0}^{C}\mathcal{D}_t^{0} h(t) = {}_t^{C}\mathcal{D}_{t_f}^{0} h(t) = h(t)$.

The fractional integrals and derivatives have the following properties (Podlubny, 1998):

$$ {}_{t_0}\mathcal{I}_t^{\gamma}\big(\zeta \cdot h(t) + \varrho \cdot q(t)\big) = \zeta \cdot {}_{t_0}\mathcal{I}_t^{\gamma} h(t) + \varrho \cdot {}_{t_0}\mathcal{I}_t^{\gamma} q(t) \tag{1a} $$

$$ {}_t\mathcal{I}_{t_f}^{\gamma}\big(\zeta \cdot h(t) + \varrho \cdot q(t)\big) = \zeta \cdot {}_t\mathcal{I}_{t_f}^{\gamma} h(t) + \varrho \cdot {}_t\mathcal{I}_{t_f}^{\gamma} q(t) \tag{1b} $$

$$ {}_{t_0}\mathcal{I}_t^{\gamma}\big({}_{t_0}^{C}\mathcal{D}_t^{\gamma} h(t)\big) = h(t) - \sum_{j=0}^{\lceil \gamma \rceil - 1} \frac{h^{(j)}(t_0)}{j!}(t-t_0)^j \tag{1c} $$

$$ {}_t\mathcal{I}_{t_f}^{\gamma}\big({}_t^{C}\mathcal{D}_{t_f}^{\gamma} h(t)\big) = h(t) - \sum_{j=0}^{\lceil \gamma \rceil - 1} \frac{(-1)^j h^{(j)}(t_f)}{j!}(t_f-t)^j \tag{1d} $$

where ζ and ϱ are constants.

Theorem 1. *There hold*

$$ {}_{t_0}\mathcal{I}_t^{\gamma} h(t) = \left(\frac{t_f - t_0}{2}\right)^{\gamma} \cdot {}_{-1}\mathcal{I}_{\tau}^{\gamma} h(\tau; t_0, t_f) \tag{2a} $$

$$ {}_{t_0}^{C}\mathcal{D}_t^{\gamma} h(t) = \left(\frac{2}{t_f - t_0}\right)^{\gamma} \cdot {}_{-1}^{C}\mathcal{D}_{\tau}^{\gamma} h(\tau; t_0, t_f) \tag{2b} $$

where

$$ t = \frac{t_f - t_0}{2}\tau + \frac{t_f + t_0}{2}, \quad \tau \in [-1, +1]. \tag{3} $$

Note that the same results also hold for the corresponding right counterparts.

Proof. The derivation is straightforward, and hence, is omitted here for brevity. □

3. Differential and integral forms of scaled FOCP

3.1. Differential form of scaled FOCP

Consider the following general FOCP. Determine the state, $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$, control, $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$, initial time, $t_0 \in \mathbb{R}$, and final time, $t_f \in \mathbb{R}$, on the time interval $t \in [t_0, t_f]$ that minimize the cost functional

$$ J = \phi(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, \mathrm{d}t, \tag{4} $$

subject to the dynamic constraints ($\gamma \in (0, 1]$)

$$ {}_{t_0}^{C}\mathcal{D}_t^{\gamma} \boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), \tag{5} $$

the inequality path constraints

$$ \boldsymbol{c}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \leq \boldsymbol{0}, \tag{6} $$

and the boundary conditions

$$ \boldsymbol{b}(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f) = \boldsymbol{0}. \tag{7} $$

Using Eqs. (3) and (2b), the FOCP of Eqs. (4)–(7) is then converted to the scaled FOCP in terms of the variable τ as follows. Determine the state, $\boldsymbol{x}(\tau) \in \mathbb{R}^{n_x}$, control, $\boldsymbol{u}(\tau) \in \mathbb{R}^{n_u}$, initial time, $t_0 \in \mathbb{R}$,