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Computing the state difference equations for discrete overdetermined linear *m*D systems☆

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ABSTRACT

We derive an algorithm that computes the state difference equations for a given set of poles of linear discrete overdetermined autonomous *m*D systems. These difference equations allow the realization of the dynamical system by means of delay, multiplication and addition elements in simulation diagrams. In doing so we generalize the classical Cayley–Hamilton theorem to multivariate polynomial ideals and provide a system theoretic interpretation to the notion of polynomial ideals, leading monomials and Gröbner bases. Furthermore, we extend the problem to include poles at infinity and so arrive at a description of overdetermined descriptor systems. This results in a new state space description of autonomous *m*D descriptor systems. In addition, we discuss the separation of the state variables of singular *m*D systems into a regular and singular part. A sufficient condition under which these two state vector parts can be interpreted as a forward evolving regular part and a backward evolving singular part is given. The robustness and efficiency of the developed algorithms are demonstrated via numerical experiments.

1. Introduction

There has been an increasing interest in multi-dimensional (*m*D) systems in the systems and control community over the past decades (Attasi, 1973; Bleylevens, Peeters, & Hanzon, 2005, 2007; Dreesen, 2013; Fornasini & Marchesini, 1976; Hazewinkel & Hanzon, 2006; Roesser, 1975). Different formulations of *m*D systems were given (Attasi, 1973; Fornasini & Marchesini, 1976; Kaczorek, 1988; Kurek, 1985; Roesser, 1975) and concepts such as realization, stability, reachability and observability have been thoroughly analysed. In the 1D case, the poles of the system determine the state recursion equation. This fact is used for the design of linear filters or for the stabilization of the linear system by state feedback (the pole placement problem). In this article we extend the theory on how to determine the state difference equations for a given set of poles to the following class of discrete overdetermined multiple input-multiple output (MIMO) *m*D linear systems

$$\begin{aligned} x(t_1+1,\ldots,t_m) &= A_1x(\mathbf{t})+B_1u(\mathbf{t}),\\ &\vdots\\ x(t_1,\ldots,t_m+1) &= A_mx(\mathbf{t})+B_mu(\mathbf{t}),\\ y(\mathbf{t}) &= C\,x(\mathbf{t})+Du(\mathbf{t}), \end{aligned} \tag{1}$$

where we introduce the shorthand notation $\mathbf{t} \triangleq (t_1,\ldots,t_m)$ and $x(\mathbf{t}) \in \mathbb{R}^n$, $u(\mathbf{t}) \in \mathbb{R}^p$ and $y(\mathbf{t}) \in \mathbb{R}^l$. The matrices $A_1,\ldots,A_m$, $B_1,\ldots,B_m$, C, D have the appropriate dimensions. Note how the updating of the whole state $x(\mathbf{t})$ is described by a separate state equation for each independent direction $t_1,\ldots,t_m$. It is for this reason that (1) are called overdetermined *m*D systems. Continuous overdetermined systems are characterized by having partial derivatives of the state with respect to $t_1,\ldots,t_m$ on the left hand side. These systems were originally studied in the field of operator theory (Livšic, Kravitsky, Markus, & Vinnikov, 1995; Rosenthal & Gilliam, 2003). More recently, the pole placement problem for state feedback of overdetermined continuous 2D systems was fully solved (Shaul & Vinnikov, 2009). Discrete overdetermined systems as described by (1) have only been studied in recent years (Bleylevens et al., 2005, 2007; Dreesen, 2013; Hazewinkel & Hanzon, 2006). They have found direct applications in model order reduction (Bleylevens et al., 2005) and

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polynomial optimization (Bleylevens et al., 2007). Furthermore, the relations between autonomous overdetermined mD systems, solving multivariate polynomial systems, Gröbner bases and systems theory are discussed in Dreesen (2013) and Hazewinkel and Hanzon (2006).

Other commonly used mD state space models are the Roesser model (Roesser, 1975) and the Fornasini–Marchesini model (Fornasini & Marchesini, 1976). It is important to observe that the state space system (1) is quite different from the Roesser model in that the state vector $x(\mathbf{t})$ is not divided into different partial state vectors along each dimension. It is also not required to have an infinite amount of initial states in order to compute the state recursion. In fact we will show that, just like in the 1D case, the system order n denotes the total number of previous states required in the state difference equations in order to compute the next state. It is possible for the 2D case to write (1) in terms of the general model of Kurek (1985), which generalizes the Roesser, Fornasini–Marchesini and Attasi models. Indeed, the general model is

$$\begin{aligned} x(t_1+1,t_2+1) = {} & G_0\,x(t_1,t_2) + G_1\,x(t_1,t_2+1) \\ & + G_2 x(t_1+1,t_2) + H_0 u(t_1,t_2) \\ & + H_1 u(t_1,t_2+1) + H_2 u(t_1+1,t_2), \end{aligned}$$

which is equivalent with (1) if $G_0 = A_1A_2 = A_2A_1$, $H_0 = (A_1B_2 + A_2B_1)/2$, $G_1 = G_2 = 0$ and $H_1 = B_1/2$, $H_2 = B_2/2$ are satisfied. However, the general model of Kurek (1985) still needs an infinite amount of initial states in order to compute all future states. Note that the A_i matrices also commute for the Attasi models (Attasi, 1973).

The main contribution of this article is the derivation and implementation of a numerical algorithm that solves the following problem:

Problem 1. Given the finite set of poles $\mathcal{Z}$ of (1), find a minimal set of difference equations $G = \{g_1, \ldots, g_k\}$ that allow us to recursively compute all future states of the corresponding autonomous mD system through monomial shifts $z_1^{t_1} \cdots z_m^{t_m}$ of $g_1, \ldots, g_k$.

Consequently, these future states completely determine the free response ($u(\mathbf{t}) \equiv 0\,\forall\, t_1, \ldots, t_m \geq 0$) of the overdetermined mD system (1). Problem 1 is trivially solved for the 1D case. Indeed, the unique monic characteristic polynomial $g_1(z_1)$ of A_1 in the shift operator z_1 is $(z_1 - \lambda_1)\cdots(z_1 - \lambda_n) = z_1^n + c_{n-1}z_1^{n-1} + \cdots + c_0$. The Cayley–Hamilton theorem allows us to write

$$g_1(A_1) = A_1^n + c_{n-1}A_1^{n-1} + \cdots + c_0 I_n = 0,$$

which can be post-multiplied with $x(0)$ to obtain

$$g_1 = x(n) + c_{n-1}x(n-1) + \cdots + c_0 x(0) = 0, \tag{2}$$

since $x(t) = A_1^t x(0)$ for an autonomous system. The linear relation (2) can be interpreted as a way to compute $x(n)$ from the known states $x(n-1), \ldots, x(0)$ without needing to make an explicit choice of a basis. Consequently, all future states $x(t)$ $(t > n)$ are found from multiplying (2) with powers of z_1 (or A_1). Indeed, $x(n+1)$ can be computed from multiplying (2) with z_1, $x(n+2)$ from multiplying (2) with z_1^2 and so forth. In other words, the set of all polynomials in z_1 from which all future states $x(t)$ can be computed form a polynomial ideal $\langle p(z_1)\rangle$. Generalizing this result to the mD case ($m > 1$) is less trivial. We will show that for the mD case the characteristic polynomial of A_1 needs to be replaced by a Gröbner basis. Essentially, solving this problem is equivalent with computing a Gröbner basis that vanishes on the given set of poles $\mathcal{Z}$. The first symbolical algorithm that solves this problem was developed by Möller and Buchberger Möller and Buchberger (1982). A numerical implementation of the affine Buchberger–Möller algorithm is described in Heldt, Kreuzer, Pokutta, and Poulisse (2009). Their numerical implementation uses both the singular value decomposition (SVD) and Gaussian Elimination (GE). We will derive our own version of both the affine and projective Buchberger–Möller algorithm, using only the SVD, from a system theoretic point of view.

Once a Gröbner basis $g_1 \ldots, g_k$ is found that vanishes on the given set of poles $\mathcal{Z}$, we then have a set of polynomials that generate the polynomial ideal $\langle g_1, \ldots, g_k\rangle$ of state difference polynomials. The generalization of the Cayley–Hamilton theorem then states that the A_i matrices of the mD system (1) satisfy all polynomial equations $p(A_1, \ldots, A_m) = 0$ with $p \in \langle g_1, \ldots, g_k\rangle$. Many generalizations of the Cayley–Hamilton theorem have appeared (Kaczorek, 2005; Kurek, 1985; Theodorou, 1989), but none from the point of view of Gröbner bases and systems theory. More theory on Gröbner bases and their application in systems theory can be found in Bose (2010) and Buchberger (2001). Poles at infinity of linear systems have been extensively studied and it is commonly known that poles at infinity are intimately linked with the notion of singular or descriptor systems (Kaczorek, 1988; Luenberger, 1978; Misra, Van Dooren, & Syrmos, 1995). These notions will also be generalized here to the mD case, leading to a new state space description of singular mD systems in terms of a homogenization variable t_0.

The inverse problem of Problem 1 has received more attention in the literature. It is the realization problem of finding a set of A_i matrices from a given set of difference equations (Fornasini, Rocha, & Zampieri, 1993) or mD data trajectories (Zerz, 2008), which have both been solved in the behavioural context. An alternative realization method that does not rely on the behavioural context, nor on the computation of a Gröbner basis, is a generalization of the Ho–Kalman realization algorithm (Ho & Kalman, 1966) to the mD case (Dreesen, 2013). This problem is trivially solved when the poles $\mathcal{Z}$ are given. Indeed, one can then construct the A_i matrices in their diagonal or upper triangular form using the method described in Batselier, Dreesen, and De Moor (2014). The problem solved in this article is precisely the inverse of the aforementioned realization problem: from a given set of poles $\mathcal{Z}$, find the minimal set of state difference equations. These state difference equations can then be easily realized with delay, multiplication and addition elements into a simulation diagram.

This article is organized as follows. In Section 2 we establish the notation and define some required concepts from algebraic geometry in the context of systems theory. In Section 3 we define regular systems and derive the algorithm to compute all state difference polynomials from a given set of poles. The conceptual changes necessary to extend the algorithm to work for singular systems are discussed in Section 4. The application of the algorithms for both regular as for singular systems are demonstrated in Section 5. All algorithms are implemented in MATLAB and Octave and are freely available at https://github.com/kbatseli/PNLA_MATLAB_OCTAVE.

2. Preliminaries

2.1. State orderings

All signals and state vectors are indexed by m independent variables $t_1, \ldots, t_m$. Hence, in order to be able to define a state sequence, one needs to introduce a total ordering on the m-tuples $\mathbf{t} \in \mathbb{N}_0^m$. For notational convenience, we introduce the shorthand notation $x(\mathbf{0})$ for the initial state $x(0, \ldots, 0)$. Then, by introducing the m linear shift operators $z_1, \ldots, z_m$ such that

$$z_k\,x(\mathbf{t}) = x(t_1, \ldots, t_k + 1, \ldots, t_m)\ (1 \leq k \leq m),$$