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Brief paper

Delayed sliding mode control ☆

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ABSTRACT

A new sliding mode control approach is introduced in this work with the dedicated mathematical tools. A time-delay modification/approximation of sign function is proposed, and it is shown that by substituting this new "sign" realization in the conventional sliding mode algorithms the main advantages of the sliding mode tools are preserved (like rejection of matched disturbances and hyper-exponential convergence, *i.e.* the rate of convergence to the origin is much faster than any exponential (Polyakov, Efimov, Perruquetti, & Richard, 2014)), while the chattering is reduced. These results are illustrated and confirmed by numerical simulations for the first order sliding mode control and the super-twisting algorithm.

1. Introduction

The sliding mode control and estimation algorithms became very popular nowadays due to their strong advantages: finite time of convergence and compensation of matched disturbances (Boiko, 2009; Fridman, 2011; Perruquetti & Barbot, 2002). Despite of that the sliding mode solutions have also several drawbacks mainly originated by the impossibility of a perfect practical realization of sliding motion. Among these shortages it is necessary to mention the chattering phenomenon, which is a high frequency oscillation of control signal when trajectories stay around the sliding surface. The appearance of chattering may physically destroy actuator and/or degrade the performance of transients (Levant, 2010; Utkin, 1992). There are several approaches for chattering reduction (Fridman, 2011). One of the most popular deals with the high order sliding mode algorithms (Edwards & Shtessel, 2014; Levant, 1993), where the discontinuity, for example

$$\mathrm{sign}[y(t)] = \begin{cases} \dfrac{y(t)}{|y(t)|}, & y(t) \neq 0 \\ 0, & y(t) = 0, \end{cases}$$

with $y(t) \in \mathbb{R}$ and $|\cdot|$ is the absolute value, appears not in the control/estimated variable itself but in its derivative of higher order (Fridman, 2011; Perruquetti & Barbot, 2002). However, since a discontinuity is present in the system closed loop, it may negatively influence the transients through the corresponding derivative (if this derivative has a physical meaning in the system). That is why frequently for practical realization of sliding mode algorithms different approximations of sign functions are used (Ambrosino, Celentano, & Garofalo, 1985; Burton & Zinober, 1986), like for instance

$$\hat{\mathrm{sign}}[y(t)] = \frac{y(t)}{\epsilon + |y(t)|}$$

with some sufficiently small $\epsilon > 0$, or see also Canudas-de Wit and Perruquetti (2002) for interesting general sigmoid and dynamic approximations. Such a control based on approximated sign functions received the name of "continuous" sliding mode (Khalil, 2002; Oza, Orlov, Spurgeon, Aoustin, & Chevallereau, 2014; Shtessel, Shkolnikov, & Brown, 2003). The main drawback of existent approximations is that a chattering reduction is achieved by a price of quality loss (appearance of static error in the presence of matched disturbances and, consequently, practical stability

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with an exponential rate of convergence Ambrosino et al., 1985, Esfandiari & Khalil, 1991). In the present work a development of the sliding mode control is presented that is based on a sign approximation using the time-delay framework, which guarantees the quality preservation (there is no static error in the presence of a matched disturbance and locally the speed of convergence is faster than any exponential).

Usually appearance of a delay in the system leads to performance degradation and complication of stability analysis. However in this work we are going to introduce delay in a proper way into the system, in order to make a modification of the sign function, proving certain performance in the system. The obtained approximation is very simple and can be easily implemented in control/estimation systems based on the sliding mode algorithms:

$$\mathrm{sign}_\tau(y_t) = \frac{y(t)}{\max\{|y(t)|, |y(t-\tau)|\}},$$

where $y_t \in C_{[-\tau,0]}$, $y(t) \in \mathbb{R}$ is a variable whose sign has to be evaluated at time instant $t \in \mathbb{R}$ and $\tau > 0$ is a fixed delay (design parameter). Obviously,

$$\mathrm{sign}_0(y_t) = \mathrm{sign}[y(t)].$$

In this work the new delayed sliding mode control framework will be substantiated, it is based on such an approximation and admits the following advantages:

- exact cancellation of matched disturbances is preserved despite of approximation;
- the control algorithms demonstrate hyper-exponential convergence to the origin;
- chattering reduction with respect to conventional sliding mode control.

The idea of chattering reduction, roughly speaking, is based on the fact that

$$\{\phi \in C_{[-\tau,0]} : \mathrm{sign}_\tau(\phi) = 0\} \subset \{\phi \in C_{[-\tau,0]} : \mathrm{sign}[\phi(0)] = 0\}.$$

The outline of this work is as follows. The preliminary definitions for time-delay systems are given in Section 2. The motivating example of the first order sliding mode control algorithm is considered in Section 3. An extension to high order sliding mode algorithms is presented in Section 4. Several examples are considered for illustration of the obtained results.

2. Preliminaries

Consider a functional differential equation of retarded type (Kolmanovsky & Nosov, 1986):

$$dx(t)/dt = f(x_t, d(t)), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in C_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ (we denote by $C_{[-\tau,0]}$, $0 < \tau < +\infty$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $\|\phi\| = \sup_{-\tau\leq\varsigma\leq 0} |\phi(\varsigma)|$, where $|\cdot|$ is the standard Euclidean norm); $d(t) \in \mathbb{R}^m$ is an essentially bounded measurable input, *i.e.* $\|d\|_\infty = ess\sup_{t\geq 0} |d(t)| < +\infty$; $f : C_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally bounded functional, $f(\mathbf{0}, 0) = 0$ where $\mathbf{0}(s) = 0$ for all $-\tau \leq s \leq 0$. The representation (1) includes pointwise or distributed time-delay systems. We consider the system (1) with the initial functional condition $x_0 \in C_{[-\tau,0]}$.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r \in \mathbb{R}_+$.

2.1. Discontinuous functional differential equations

The dynamical systems subjected by a time-delay, whose models are described by functional differential equations, find their applications in different areas of science and technology (Chiasson & Loiseau, 2007). Analysis of delay influence on the system stability is critical for many natural and human-developed systems (Gu, Kharitonov, & Chen, 2003; Kolmanovsky & Nosov, 1986; Richard, 2003). The problem of stability investigation in time-delay systems is much more complicated than for ordinary differential equations since design of a Lyapunov–Krasovskii functional or a Lyapunov–Razumikhin function is a complex issue.

It is known from the theory of functional differential equations (Kolmanovsky & Nosov, 1986) that under the above assumptions the system (1) with a locally Lipschitz f has a unique solution $x(t, x_0, d)$ satisfying the initial condition x_0 for the input $d(t)$, which is defined on some finite time interval $[-\tau, T)$ (we will use the notation $x(t)$ to reference $x(t, x_0, d)$ if the origin of x_0 and d is clear from the context). If function $f(\phi, d) \equiv f_0(\phi(0), \phi(-\tau), d)$, where $f_0 : \mathbb{R}^{2n+m} \to \mathbb{R}^n$, and it is discontinuous with respect to $\phi(0)$ on a set $\mathcal{N} \subset \mathbb{R}^n$ of measure zero only and continuous with respect to $\phi(-\tau)$ and d, following Filipov (1988) and Heemels and Weiland (2008) (see also Kolmanovskii & Myshkis, 1999, Surkov, 2008 for a general definition and existence conditions of solutions for discontinuous functional differential equations) we will consider its multi-valued extension (define $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x-y| \leq \varepsilon\}$ as a closed ball of radius $\varepsilon > 0$ around $x \in \mathbb{R}^n$):

$$F(\phi, d) = \bigcap_{\varepsilon>0} \overline{\mathrm{conv}}[f_0\{B_\varepsilon(\phi(0)) \setminus \mathcal{N}, \phi(-\tau), d\}],$$

which is non-empty, compact, convex and upper semi-continuous (Heemels & Weiland, 2008) for any $d \in \mathbb{R}^m$. In particular, the multi-valued extension of $\mathrm{sign}_\tau(y_t)$ can be defined as follows:

$$\overline{\mathrm{sign}}_\tau(y_t) = \begin{cases} [-1, 1], & y(t) = y(t-\tau) = 0 \\ \dfrac{y(t)}{\max\{|y(t)|, |y(t-\tau)|\}}, & \text{otherwise.} \end{cases}$$

In this case, instead of (1), the solutions of the following functional differential inclusion will be considered:

$$dx(t)/dt \in F(x_t, d(t)), \quad t \geq 0, \tag{2}$$

and for any initial condition $x_0 \in C_{[-\tau,0]}$ the set of corresponding solutions of (2) initiated at x_0 for the input d can be denoted as $\mathcal{S}(x_0, d)$.

Remark 1. Considering solutions of the system (1) for $f(\phi, d) \equiv f_0(\phi(0), \phi(-\tau), d)$ on the interval $[0, \tau]$ we can derive the Filippov differential inclusion for $\tilde{f}_0[t, x(t), d(t)] = f_0[x(t), x_0(t-\tau), d(t)]$ on the right-hand side. The obtained inclusion satisfies all conditions of existence theorem (Filipov, 1988) implying existence of a solution locally. Having the solution defined on $[0, \tau]$ we can repeat the same considerations for $t \in [\tau, 2\tau]$, *etc.* This method of steps allows solutions of (2) to be defined for $t > 0$.

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ (where $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$) let us introduce the upper directional Dini derivative along the trajectories of (2):

$$D^+_{F(x_t,d)} V[x_t(0)] = \sup_{v\in F(x_t,d)} \limsup_{h\to 0^+} \frac{V[x_t(0) + hv] - V[x_t(0)]}{h}.$$