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Brief paper

Stability of nonlinear differential systems with state-dependent delayed impulses☆

Xiaodi Li[a,b], Jianhong Wu[b]

[a] School of Mathematical Sciences, Shandong Normal University, Ji'nan, 250014, PR China

[b] Laboratory for Industrial and Applied Mathematics, York University, Toronto, Ontario, Canada, M3J 1P3

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ABSTRACT

We consider nonlinear differential systems with state-dependent delayed impulses (impulses which involve the delayed state of the system for which the delay is state-dependent). Such systems arise naturally from a number of applications and the stability issue is complex due to the state-dependence of the delay. We establish general and applicable results for uniform stability, uniform asymptotic stability and exponential stability of the systems by using the impulsive control theory and some comparison arguments. We show how restrictions on the change rates of states and impulses should be imposed to achieve system's stability, in comparison with general impulsive delay differential systems with state-dependent delay in the nonlinearity, or the differential systems with constant delays. In our approach, the boundedness of the state-dependent delay is not required but derives from the stability result obtained. Examples are given to demonstrate the sharpness and applicability of our general results and the proposed approach.

1. Introduction

Impulsive delay differential systems have been used for modelling natural phenomena in many areas for many years, and there have been significant studies of such systems, as indicated by Churilov and Medvedev (2014), Dashkovskiy, Kosmykov, Mironchenko, and Naujok (2012), Lakshmikantham, Bainov, and Simeonov (1989), Li, Bohner, and Wang (2015), Sakthivel, Mahmudov, and Kim (2009), Sakthivel, Ren, and Mahmudov (2010) and Samoilenko and Perestyuk (1995) and references therein. Of current interest is the delayed impulses of differential systems arising in such applications as automatic control, secure communication and population dynamics (Akca, Alassar, Covachev, Covacheva, & Al-Zahrani, 2004; Akhmet & Yilmaz, 2014; Chen, Wei, & Lu, 2013; Chen & Zheng, 2011, 2009; Khadra, Liu, & Shen, 2005, 2009; Liu, Teo, & Xu, 2005), here and in what follows, *a delayed impulse* describes a phenomenon where impulsive transients depend on not only their current but also historical states of the system. For instance, in communication security systems based on impulsive synchronization, there exist transmission and sampling delays during the information transmission process, where the sampling delay created from sampling the impulses at some discrete instances causes the impulsive transients depend on their historical states (Chen et al., 2013; Khadra et al., 2005). The existing studies, however, such as those in Akca et al. (2004), Akhmet and Yilmaz (2014), Chen et al. (2013), Chen and Zheng (2011, 2009), Khadra et al. (2005, 2009) and Liu et al. (2005), assume the delays in impulsive perturbations are either fixed as constants or given by integrals with state-independent distributed kernels. For example, Khadra et al. (2005) considered the impulsive synchronization of chaotic systems with transmission delay and sampling delay, and then applied the results to the design of communication security scheme. Chen and Zheng (2011) studied the nonlinear time-delay systems with two kinds of delayed impulses, that is, destabilizing delayed impulses and stabilizing delayed impulses, and derived some interesting results for exponential stability. But in both results, the delays in impulses are given constants. Akca et al. (2004) derived some results for global stability of Hopfield-type neural networks with delayed impulses, where the delays in impulses are in integral forms with state-independent distributed kernels. However, in many cases it is important to consider state-dependent delays in impulsive perturbations. For example, the sampling delay varies with the change of state variables since it is natural to consider sending control signals less frequently when

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E-mail addresses: sodymath@163.com (X. Li), wujh@yorku.ca (J. Wu).

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the state is small and more frequently when the state is large (Hespanha, Naghshtabrizi, & Xu, 2007); while in some other impulsive models arising from disease control, financial options, and population dynamics, it is also natural to introduce state-dependent delays into the impulses. There have already been some results in the literature on the existence, uniqueness and controllability for some classes of differential systems with impulses involving state-dependent delay (see Chang, Nieto, & Zhao, 2010, Hernandez, Sakthivel, & Aki, 2008, Liu & Ballinger, 2001, Sakthivel & Anandhi, 2010), but little seems to have been done for the stability.

We should have mentioned that state-dependent delay incorporated in the differential system has also found increasing applications in a variety of fields, such as control systems (Hespanha et al., 2007; Liberis & Krstic, 2013b; Niemeyer & Slotine, 2001), turning processes (Insperger, Stepan, & Turi, 2007), complex networks (Sterman, 2000; Witrant, Carlos, Georges, & Alamir, 2007), and biological systems (Adimy, Crauste, Hbid, & Qesmi, 2010; Aiello, Freedman, & Wu, 1992). Many interesting and important results for state-dependent delay systems have been recently reported (see Ecimovic & Wu, 2002, Hartung, Krisztin, Walther, & Wu, 2006, Paret & Nussbaum, 2011, Sakthivel & Ren, 2013, Walther, 2008 and references therein) including stability analysis (Cooke & Huang, 1996; Gyori & Hartung, 2007; Hartung & Turi, 1995; Liberis & Krstic, 2013a; Verriest, 2002). However, many classical methods for stability analysis of delay systems, including delay decomposition approach, free-weighting matrix method, and Leibniz–Newton formula have not been extended to differential systems with state-dependent delay in general, and differential systems with state-dependent delayed impulses in particular.

In this study, we focus on stability problem of nonlinear differential systems with impulses involving state-dependent delay based on Lyapunov methods. As is well known, in systems with time delays, there exist two main Lyapunov methods for stability analysis: the Krasovskii method of Lyapunov functionals and the Razumikhin method of Lyapunov functions. However, when the time delays exist in impulses and moreover is state-dependent, there are substantial difficulties to apply either method. In fact, due to the existence of state delay in impulses, it is hard to know exactly a priori how far in the history the information is needed, and is hard to determine the historical states at impulsive instances. Moreover, it is possible that function V along a solution can be increasing at certain impulses points due to the state-dependence of the delay. In this study, we provide some new insights on the features of systems with impulses involving state-dependent delay, and give an estimate of Lyapunov functions which is coupled with the effect of state delay based on impulsive control theory and some comparison arguments. Then we establish (in Section 3) some general results for (Lyapunov) uniform stability, uniform asymptotic stability and exponential stability, where the necessary constraint on state-dependent delay is specified of boundedness of the state-dependent delay is not required a priori. We will also provide, in Section 4, numerical examples to demonstrate the effectiveness of the proposed approach and our established results.

2. Preliminaries

Notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the n-dimensional and $n \times m$-dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$, respectively, $\mathbb{Z}_+$ the set of positive integer numbers, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the maximum and minimum eigenvalues of symmetric matrix A, respectively. $A > 0$ or $A < 0$ denotes that the matrix A is a symmetric and positive or negative definite matrix. I the identity matrix with appropriate dimensions. $\mathscr{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) |\ a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0 \text{ and } a \text{ is increasing in } s\}$.

Consider the following impulsive differential system

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \quad t \ge t_0 \ge 0,\ t \ne t_k, \\ x(t_k) = I_k(t_k^- - \tau, x(t_k^- - \tau)), \quad \tau = \tau(t_k, x(t_k^-)), \\ x_{t_0} = \phi, \end{cases} \tag{1}$$

where $\phi \in \mathbb{C}_\alpha$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $I_k \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $k \in \mathbb{Z}_+$, $\tau \in C(\mathbb{R}_+ \times \mathbb{R}^n, [0, \alpha])$, $x_{t_0} = \{x(t_0 + s) : s \in [-\alpha, 0]\}$, $0 \le \alpha \le +\infty$, especially when $\alpha = \infty$, the interval $[s - \alpha, s]$ is understood to be replaced by $(-\infty, s]$ for any $s \in \mathbb{R}$. $\mathbb{C}_\alpha \doteq C([-\alpha, 0], \mathbb{R}^n) = \{\phi : [-\alpha, 0] \to \mathbb{R}^n \text{ is continuous}\}$ with the norm $\|\phi\|_\alpha = \sup_{-\alpha\le\theta\le 0} \|\phi(\theta)\|$ for $\phi \in \mathbb{C}_\alpha$. Given a constant $\mathcal{M} > 0$, set $\mathbb{C}_\alpha^{\mathcal{M}} = \{\phi \in \mathbb{C}_\alpha : 0 < \|\phi\| \le \mathcal{M}\}$. The impulse times t_k satisfy $0 \le t_0 < t_1 < \cdots < t_k \to +\infty$ as $k \to \infty$.

Note that the continuity of f, I_k and τ, and a fact that system (1) is an ODE which is continuous on each interval $[t_{k-1}, t_k)$. We assume that the vector field f satisfies suitable conditions so the solutions exist in relevant time intervals. These conditions can be formulated using standard conditions such as conditions (H_1)–(H_3) in Liu and Ballinger (2001) (or Lakshmikantham et al., 1989). Denote by $x(t) \doteq x(t, t_0, \phi)$ the solution of the system (1). In addition, we always assume that $f(t, 0) \equiv 0$, $t \ge t_0$, and $I_k(t, x) = 0$ if and only if $x = 0$, $t \ge t_0$, $k \in \mathbb{Z}_+$. Thus system (1) admits a trivial solution $x(t) \equiv 0$. Some definitions are given in the following.

Definition 1. The function $V : [t_0 - \alpha, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to class υ_0 if

(1) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and $\lim_{(t,u)\to(t_k^-,v)} V(t, u) = V(t_k^-, v)$ exists;
(2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2. Let $V \in \upsilon_0$, D^+V is defined as

$$D^+V(t, x(t)) = \limsup_{h\to 0^+} \frac{1}{h}\Big\{V(t + h, x(t) + hf(t, x(t))) - V(t, x(t))\Big\}.$$

Definition 3. System (1) is said to be

(1) locally uniformly stable (LUS) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, if there exists a constant $\mathcal{M} > 0$, and if for any $t_0 \ge 0$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, \mathcal{M}) \in (0, \mathcal{M}]$ such that $\phi \in \mathbb{C}_\alpha^\delta$ implies $|x(t, t_0, \phi)| < \varepsilon$, $t \ge t_0$;
(2) locally uniformly asymptotically stable (LUAS) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, if it is uniformly stable and uniformly attractive;
(3) locally exponentially stable (LES) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, if there exist constants $\lambda > 0$, $M^\star \ge 1$, $\mathcal{M} > 0$ such that
$$\|x(t)\| \le M^\star \|\phi\|_\alpha\, e^{-\lambda(t-t_0)}, \quad t \ge t_0,$$
for any initial value $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

3. Main results

Theorem 1. *Assume that there exist constants $\gamma > 0$, $\theta \in (0, 1)$, $\mathcal{M} > 0$, $\rho_k \ge 1$, $k \in \mathbb{Z}_+$ functions $\omega_1, \omega_2 \in \mathscr{K}$, $H \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and $V \in \upsilon_0$ such that*

(i) $\omega_1(\|x\|) \le V(t, x) \le \omega_2(\|x\|)$ *for all* $(t, x) \in [t_0 - \alpha, \infty) \times \mathbb{R}^n$;
(ii) $D^+V(t, x(t)) \le -H\Big(t, V(t, x(t))\Big)$, $t \in [t_{k-1}, t_k)$;
(iii) $V(t_k, x(t_k)) \le \rho_k V(t_k^- - \tau, x(t_k^- - \tau))$, $\tau = \tau(t_k, x(t_k^-))$, $k \in \mathbb{Z}_+$, *where* $x(t) = x(t, t_0, \phi)$ *is a solution of* (1);
(iv) $|\tau(s, \mathbf{u}) - \tau(s, \mathbf{0})| \le \gamma\|u\|$ *for any* $s \in \mathbb{R}_+$, $\mathbf{u} \in \mathbb{R}^n$;
(v) $\tau^\star \doteq \sup_{t\ge t_0} \tau(t, \mathbf{0}) < \infty$;