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Brief paper

Value function in maximum hands-off control for linear systems☆

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ABSTRACT

In this brief paper, we study the value function in maximum hands-off control. Maximum hands-off control, also known as sparse control, is the L^0-optimal control among the feasible controls. Although the L^0 measure is discontinuous and non-convex, we prove that the value function, or the minimum L^0 norm of the control, is a continuous and strictly convex function of the initial state in the reachable set, under an assumption on the controlled plant model. We then extend the finite-horizon maximum hands-off control to model predictive control (MPC), and prove the recursive feasibility and the stability by using the continuity and convexity properties of the value function.

1. Introduction

Optimal control is widely used in recent industrial products not just for achieving the best performance but for reducing the control effort. For example, the classical LQR (Linear Quadratic Regulator) control gives a way to consider the tradeoff between performance and control-effort reduction by using weighting functions on the states and the control inputs with the L^2 norm (i.e. the energy); see Anderson and Moore (2007) for example.

Recently, a novel control method, called *maximum hands-off control*, has been proposed in Nagahara, Quevedo, and Nešić (2013, 2016), which maximizes the time duration in which the control is exactly zero among the feasible controls. An example of hands-off control is a stop–start system in automobiles, in which an automobile automatically shuts down the engine (i.e. zero control) to avoid it idling for long periods of time, and also to reduce CO or CO_2 emissions as well as fuel consumption. Therefore, the hands-off control is a kind of *green control* as discussed in Nagahara, Quevedo, and Nešić (2014b). Also, the hands-off control is effective in hybrid/electric vehicles, railway vehicles, networked/embedded systems, to name a few; see Nagahara et al. (2016).

Maximum hands-off control is related to *sparsity*, which is widely studied in compressed sensing, for which we point the reader to Eldar and Kutyniok (2012). Sparsity is also applied to control problems such as networked control in Kong, Goodwin, and Seron (2015) and Nagahara, Quevedo, and Østergaard (2014a), security of control systems in Fawzi, Tabuada, and Diggavi (2014), state estimation in Sanandaji, Wakin, and Vincent (2014), to name a few.

A mathematical difficulty in the maximum hands-off control is that the cost function, which is defined by the L^0 measure (the support length of a function), is highly nonlinear; it is discontinuous and non-convex. To solve this problem, a recent work of Nagahara et al. (2013, 2016) has proposed to reduce the problem to an L^1 optimal control problem, and shown the equivalence between the maximum hands-off (or L^0 optimal) control and the L^1 optimal control under the assumption of normality.

Motivated by this work, we investigate the value function in the maximum hands-off control. The value function is defined as the optimal value of the cost function of the optimal control problem. Although the L^0 measure in the maximum hands-off control is discontinuous and non-convex, we prove that the value function is a continuous and strictly convex function of the initial state in the reachable set, under an assumption on the controlled plant model. We then extend the finite-horizon maximum hands-off control to model predictive control (MPC), and prove the recursive feasibility (see Rossiter (2004)) and the stability by using the continuity and convexity properties of the value function.

The present paper expands on our recent conference contributions of Ikeda and Nagahara (2015a,b) by rearranging the contents,

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incorporating convexity analysis of the value function, and including extension to model predictive control.

The remainder of this paper is organized as follows: In Section 2, we give mathematical preliminaries for our subsequent discussion. In Section 3, we review the problem of maximum hands-off control. Section 4 investigates the continuity of the value function in maximum hands-off control, and Section 5 discusses its convexity. Section 6 discusses model predictive control and the stability. Section 7 presents an example of model predictive control to illustrate the effectiveness of the proposed method. In Section 8, we offer concluding remarks.

2. Mathematical preliminaries

This section reviews basic definitions, facts, and notation that will be used throughout the paper.

Let n be a positive integer. For a vector $x \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$, the *ε-neighborhood* of x is defined by

$$\mathcal{B}(x, \varepsilon) \triangleq \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Let $\mathcal{X}$ be a subset of $\mathbb{R}^n$. A point $x \in \mathcal{X}$ is called an *interior point* of $\mathcal{X}$ if there exists $\varepsilon > 0$ such that $\mathcal{B}(x, \varepsilon) \subset \mathcal{X}$. The *interior* of $\mathcal{X}$ is the set of all interior points of $\mathcal{X}$, and we denote the interior of $\mathcal{X}$ by $\operatorname{int}\mathcal{X}$. A set $\mathcal{X}$ is said to be *open* if $\mathcal{X} = \operatorname{int}\mathcal{X}$. For example, $\operatorname{int}\mathcal{X}$ is open for every subset $\mathcal{X} \subset \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is called an *adherent point* of $\mathcal{X}$ if $\mathcal{B}(x, \varepsilon) \cap \mathcal{X} \neq \emptyset$ for every $\varepsilon > 0$, and the *closure* of $\mathcal{X}$ is the set of all adherent points of $\mathcal{X}$. A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *closed* if $\mathcal{X} = \overline{\mathcal{X}}$, where $\overline{\mathcal{X}}$ is the closure of $\mathcal{X}$. The *boundary* of $\mathcal{X}$ is the set of all points in the closure of $\mathcal{X}$, not belonging to the interior of $\mathcal{X}$, and we denote the boundary of $\mathcal{X}$ by $\partial\mathcal{X}$, i.e., $\partial\mathcal{X} = \overline{\mathcal{X}} - \operatorname{int}\mathcal{X}$, where $\mathcal{X}_1 - \mathcal{X}_2$ is the set of all points which belong to the set $\mathcal{X}_1$ but not to the set $\mathcal{X}_2$. In particular, if $\mathcal{X}$ is closed, then $\partial\mathcal{X} = \mathcal{X} - \operatorname{int}\mathcal{X}$, since $\mathcal{X} = \overline{\mathcal{X}}$. A set $\mathcal{X} \subset \mathbb{R}^n$ is said to be *convex* if, for any $x, y \in \mathcal{X}$ and any $\lambda \in [0, 1]$, $(1 - \lambda)x + \lambda y$ belongs to $\mathcal{X}$.

A real-valued function f defined on $\mathbb{R}^n$ is said to be *upper semi-continuous* on $\mathbb{R}^n$ if for every $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : f(x) < \alpha\}$ is open, and f is said to be *lower semi-continuous* on $\mathbb{R}^n$ if for every $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : f(x) > \alpha\}$ is open. It is known that a function f is continuous on $\mathbb{R}^n$ if and only if it is upper and lower semi-continuous on $\mathbb{R}^n$; see e.g. Rudin (1987, pp. 37). A real-valued function f defined on a convex set $\mathcal{C} \subset \mathbb{R}^n$ is said to be *convex* if

$$f\big((1 - \lambda)x + \lambda y\big) \leq (1 - \lambda)f(x) + \lambda f(y),$$

for all $x, y \in \mathcal{C}$ and all $\lambda \in (0, 1)$, and f is said to be *strictly convex* if the above inequality holds strictly whenever x and y are distinct points and $\lambda \in (0, 1)$.

Let $T > 0$. For a continuous-time signal $u(t)$ over a time interval $[0, T]$, we define its *L^1 and L^∞ norms* respectively by

$$\|u\|_1 \triangleq \int_0^T |u(t)|dt, \qquad \|u\|_\infty \triangleq \sup_{t\in[0,T]} |u(t)|.$$

We define the support set of u, denoted by $\operatorname{supp}(u)$, by the set $\{t \in [0, T] : u(t) \neq 0\}$. The *$L^0$ norm* of a measurable function u as the length of its support, that is,

$$\|u\|_0 \triangleq m\big(\operatorname{supp}(u)\big),$$

where m is the Lebesgue measure on $\mathbb{R}$.

3. Maximum hands-off control problem

In this paper, we consider a linear time-invariant system represented by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

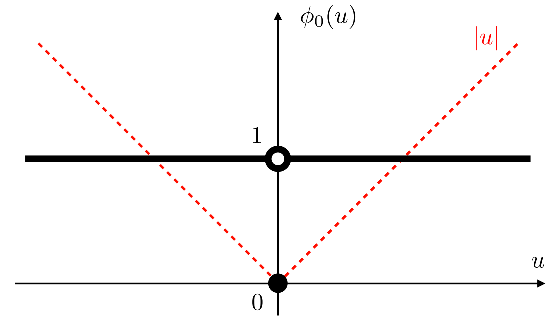

Fig. 1. The L^0 kernel $\phi_0(u)$ and its convex approximation $|u|$ for the L^1 norm.

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times 1}$. We here consider a single-input case for simplicity (see Ikeda and Nagahara (2015b) for a multi-input case). Throughout this paper, we assume the following:

Assumption 1. The pair (A, B) is controllable and the matrix A is nonsingular.

Let $T > 0$ be the final time of control. For the system (1), we call a control $u = \{u(t) : t \in [0, T]\} \in L^1$ *feasible* if it steers $x(t)$ from a given initial state $x(0) = \xi \in \mathbb{R}^n$ to the origin at time T (i.e., $x(T) = 0$), and satisfies the magnitude constraint $\|u\|_\infty \leq 1$. We denote by $\mathcal{U}(\xi)$ the set of all feasible controls for an initial state $\xi \in \mathbb{R}^n$, that is,

$$\mathcal{U}(\xi) \triangleq \left\{u \in L^1 : \int_0^T e^{-As}Bu(s)ds = -\xi,\ \|u\|_\infty \leq 1\right\}. \tag{2}$$

The *maximum hands-off control* is the minimum L^0-norm (or the sparsest) control among the feasible control inputs. This control problem is formulated as follows.

Problem 2 (*Maximum Hands-Off Control*)**.** For a given initial state $\xi \in \mathbb{R}^n$, find a feasible control $u \in \mathcal{U}(\xi)$ that minimizes $J(u) = \|u\|_0$.

The value function for this optimal control problem is defined as

$$V(\xi) \triangleq \min_{u\in\mathcal{U}(\xi)} J(u) = \min_{u\in\mathcal{U}(\xi)} \|u\|_0. \tag{3}$$

Note that the cost function $J(u)$ can be rewritten as

$$J(u) = \int_0^T \phi_0(u)\, dt,$$

where ϕ_0 is the L^0 kernel function defined by

$$\phi_0(u) \triangleq \begin{cases} 1, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Fig. 1 shows the graph of $\phi_0(u)$. As shown in this figure, the kernel function $\phi_0(u)$ is discontinuous at $u = 0$ and non-convex. However, in the following sections, we will show that the value function $V(\xi)$ in (3) is continuous and strictly convex.

4. Continuity of value function

In this section, we investigate the continuity of the value function $V(\xi)$ in (3).

First, we define the *reachable set* for the control problem (Problem 2) by

$$\mathcal{R} \triangleq \left\{\int_0^T e^{-As}Bu(s)ds : \|u\|_\infty \leq 1\right\} \subset \mathbb{R}^n.$$

The following is a fundamental lemma of the paper: