Automatica 64 (2016) 270–277

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# Generalized switching signals for input-to-state stability of switched systems[☆]

Atreyee Kundu[a,1], Debasish Chatterjee[b], Daniel Liberzon[c]

[a] Control Systems Technology Group, Department of Mechanical Engineering, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands

[b] Systems & Control Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

[c] Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

ARTICLE INFO

*Article history:*
Received 6 February 2015
Received in revised form
10 July 2015
Accepted 23 October 2015
Available online 7 December 2015

*Keywords:*
Switched systems
Input-to-state stability
Multiple-ISS Lyapunov functions

ABSTRACT

This article deals with input-to-state stability (ISS) of continuous-time switched nonlinear systems. Given a family of systems with exogenous inputs such that not all systems in the family are ISS, we characterize a new and general class of switching signals under which the resulting switched system is ISS. Our stabilizing switching signals allow the number of switches to grow faster than an affine function of the length of a time interval, unlike in the case of average dwell time switching. We also recast a subclass of average dwell time switching signals in our setting and establish analogs of two representative prior results.



## 1. Introduction

A *switched system* comprises of two components—a family of systems and a switching signal. The *switching signal* selects an active subsystem at every instant of time, i.e., the system from the family that is currently being followed (Liberzon, 2003, Section 1.1.2). Stability of switched systems is broadly classified into two categories—*stability under arbitrary switching* (Liberzon, 2003, Chapter 2) and *stability under constrained switching* (Liberzon, 2003, Chapter 3). In the former category, conditions on the family of systems are identified such that the resulting switched system is stable under all admissible switching signals; in the latter category, given a family of systems, conditions on the switching signals are identified such that the resulting switched system is stable. In this article our focus is on stability of switched systems with exogenous inputs under constrained switching.

Prior study in the direction of stability under constrained switching primarily utilizes the concept of *slow switching* vis-a-vis *(average) dwell time switching*. Exponential stability of a switched linear system under *dwell time switching* was studied in Morse (1996). In Xie, Wen, and Li (2001) the authors showed that a switched nonlinear system is ISS under dwell time switching if all subsystems are ISS. A class of state-dependent switching signals obeying dwell time property under which a switched nonlinear system is integral input-to-state stable (iISS) was proposed in De Persis, De Santis, Morse (2003). The dwell time requirement for stability was relaxed to *average dwell time switching* to switched linear systems with inputs and switched nonlinear systems without inputs in Hespanha and Morse (1999). ISS of switched nonlinear systems under average dwell time was studied in Vu, Chatterjee, and Liberzon (2007). It was shown that if the individual subsystems are ISS and their ISS-Lyapunov functions satisfy suitable conditions, then the switched system has the ISS, exponentially-weighted ISS, and exponentially-weighted iISS properties under switching signals obeying sufficiently large average dwell time. Given a family of systems such that not all systems in the family are ISS, it was shown in the recent work (Yang & Liberzon, 2014) that it is possible to construct a class of hybrid Lyapunov functions to guarantee ISS of the switched system provided that the switching signal neither switches too frequently nor activates the non-ISS subsystems for too long. In Müller and Liberzon (2012) input/output-to-state stability (IOSS) of switched nonlinear systems with families in which not all subsystems are IOSS, was studied. It was shown that the switched system is IOSS under a class of switching signals obeying *average dwell time* property and constrained point-wise activation of unstable subsystems.

Given a family of systems, possibly containing non-ISS dynamics, in this article we study ISS of switched systems under

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Graziano Chesi under the direction of Editor Richard Middleton.

*E-mail addresses:* a.kundu@tue.nl (A. Kundu), dchatter@iitb.ac.in (D. Chatterjee), liberzon@illinois.edu (D. Liberzon).

[1] Tel.: +31 40 247 4850; fax: +31 40 243 4970.

http://dx.doi.org/10.1016/j.automatica.2015.11.027

switching signals that transcend beyond the average dwell time regime in the sense that the number of switches on every interval of time can grow faster than an affine function of the length of the interval. Our characterization of stabilizing switching signals involves pointwise constraints on the duration of activation of the ISS and non-ISS systems, and the number of occurrences of the admissible switches, certain pointwise properties of the quantities defining the above constraints, and a summability condition. In particular, our contributions are:

- We allow non-ISS systems in the family and identify a class of switching signals under which the resulting switched system is ISS.
- Our class of stabilizing switching signals encompasses the average dwell time regime in the sense that on every interval of time the number of switches is allowed to grow faster than an affine function of the length of the interval. Earlier in Kundu and Chatterjee (2015) we proposed a class of switching signals beyond the average dwell time regime for global asymptotic stability (GAS) of continuous-time switched nonlinear systems.
- Although this is not the first instance when non-ISS subsystems are considered (see e.g., Müller & Liberzon, 2012 and Yang & Liberzon, 2014), to the best of our knowledge, this is the first instance when non-ISS subsystems are considered *and* the proposed class of stabilizing switching signals goes beyond the average dwell time condition.
- We recast a subclass of average dwell time switching signals in our setting and establish analogs of an ISS version of Müller and Liberzon (2012, Theorem 2), and Vu et al. (2007, Theorem 3.1) as two corollaries of our main result.

The remainder of this article is organized as follows: In Section 2 we formulate the problem under consideration and catalog certain properties of the family of systems and the switching signal. Our main results appear in Section 3, and we provide a numerical example illustrating our main result in Section 4. In Section 5 we recast prior results in our setting. The proofs of our main results are presented in a consolidated manner in Section 7.

**Notations**: Let $\mathbb{R}$ denote the set of real numbers, $\|\cdot\|$ denote the Euclidean norm, and for any interval $I \subset [0,+\infty[$ we denote by $\|\cdot\|_I$ the essential supremum norm of a map from $I$ into some Euclidean space. For measurable sets $A \subset \mathbb{R}$ we let $|A|$ denote the Lebesgue measure of $A$.

## 2. Preliminaries

We consider the *switched system*

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t), v(t)\big), \qquad x(0) = x_0 \text{ (given)}, \quad t \geq 0 \tag{1}$$

generated by

- a family of continuous-time systems with exogenous inputs
  $$\dot{x}(t) = f_i\big(x(t), v(t)\big), \qquad x(0) = x_0 \text{ (given)}, \quad i \in \mathcal{P},\ t \geq 0, \tag{2}$$
  where $x(t) \in \mathbb{R}^d$ is the vector of states and $v(t) \in \mathbb{R}^m$ is the vector of inputs at time $t$, $\mathcal{P} = \{1, 2, \ldots, N\}$ is a finite index set, and
- a piecewise constant function $\sigma : [0,+\infty[\longrightarrow \mathcal{P}$ that selects, at each time $t$, the index of the active system from the family (2); this function $\sigma$ is called a *switching signal*. By convention, $\sigma$ is assumed to be continuous from right and having limits from the left everywhere, and we call such switching signals admissible. We let $\mathcal{S}$ denote the set of all such admissible switching signals.

We assume that for each $i \in \mathcal{P}$, $f_i$ is locally Lipschitz, and $f_i(0,0) = 0$. Let the exogenous inputs $t \mapsto v(t)$ be Lebesgue measurable and essentially bounded; therefore, a solution to the switched system (1) exists in the Carathéodory sense (Filippov, 1988, Chapter 2) for some non-trivial time interval containing 0. Given a family of systems (2), our focus is on identifying a class of switching signals $\sigma \in \mathcal{S}$ under which the switched system (1) is ISS. Recall that

**Definition 1** (*Vu et al., 2007, Section 2*)**.** The switched system (1) is input-to-state stable (ISS) for a given $\sigma$ if there exist class $\mathcal{K}_\infty$ functions $\alpha$, $\chi$ and a class $\mathcal{KL}$ function $\beta$ such that for all inputs $v$ and initial states $x_0$, we have[2]

$$\alpha(\|x(t)\|) \leq \beta(\|x_0\|, t) + \chi(\|v\|_{[0,t]}) \quad \text{for all } t \geq 0. \tag{3}$$

If one can find $\alpha$, $\beta$ and $\chi$ such that (3) holds over a class $\mathcal{S}'$ of $\sigma$, then we say that (1) is uniformly ISS over $\mathcal{S}'$.

Note that when the input is set to 0, i.e., $v \equiv 0$, then (3) reduces to GAS of (1). We next catalog certain properties of the family of systems (2) and the switching signal $\sigma$. These properties will be required for our analysis towards deriving the class of stabilizing switching signals.

### 2.1. Properties of the family of systems

Let $\mathcal{P}_S$ and $\mathcal{P}_U \subset \mathcal{P}$ denote the sets of indices of ISS and non-ISS systems in the family (2), respectively, $\mathcal{P} = \mathcal{P}_S \sqcup \mathcal{P}_U$. Let $E(\mathcal{P})$ be the set of all ordered pairs $(i, j)$ such that it is admissible to switch from system $i$ to system $j$, $i, j \in \mathcal{P}$.

**Assumption 1.** There exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, $\gamma$, continuously differentiable functions $V_i : \mathbb{R}^d \to [0,+\infty[$, $i \in \mathcal{P}$, and constants $\lambda_i \in \mathbb{R}$ with $\lambda_i > 0$ for $i \in \mathcal{P}_S$ and $\lambda_i < 0$ for $i \in \mathcal{P}_U$, such that for all $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^m$, we have

$$\underline{\alpha}(\|\xi\|) \leq V_i(\xi) \leq \overline{\alpha}(\|\xi\|), \tag{4}$$

$$\left\langle \frac{\partial V_i}{\partial \xi}(\xi), f_i(\xi, \eta) \right\rangle \leq -\lambda_i V_i(\xi) + \gamma(\|\eta\|). \tag{5}$$

**Remark 1.** Conditions (4) and (5) are equivalent to an ISS version of Müller and Liberzon (2012, (7) and (18)). The functions $V_i$'s are called the *ISS-Lyapunov-like functions*, see Angeli and Sontag (1999), Krichman, Sontag, and Wang (2001) and Sontag and Wang (1995) for detailed discussion regarding the existence of such functions and their properties. In particular, condition (5) is equivalent to the ISS property for ISS subsystems (Sontag & Wang, 1995) and the unboundedness observability property for the non-ISS subsystems (Krichman et al., 2001).

**Assumption 2.** For each pair $(i, j) \in E(\mathcal{P})$ there exist $\mu_{ij} > 0$ such that the ISS-Lyapunov-like functions are related as follows:

$$V_j(\xi) \leq \mu_{ij} V_i(\xi) \quad \text{for all } \xi \in \mathbb{R}^d. \tag{6}$$

**Remark 2.** The assumption of linearly comparable Lyapunov-like functions, i.e., there exists $\mu \geq 1$ such that

$$V_j(\xi) \leq \mu V_i(\xi) \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } i, j \in \mathcal{P} \tag{7}$$

is standard in the theory of stability under average dwell time switching (Liberzon, 2003, Theorem 3.2); (6) affords sharper estimates compared to (7).

[2] $\mathcal{K} := \{\phi : [0,+\infty[\to [0,+\infty[ \,|\, \phi \text{ is continuous, strictly increasing}, \phi(0) = 0\}$, $\mathcal{KL} := \{\phi : [0,+\infty[^2 \to [0,+\infty[ \,|\, \phi(\cdot, s) \in \mathcal{K} \text{ for each } s \text{ and } \phi(r,\cdot) \searrow 0 \text{ as } s \nearrow +\infty \text{ for each } r\}$, $\mathcal{K}_\infty := \{\phi \in \mathcal{K} \,|\, \phi(r) \to +\infty \text{ as } r \to +\infty\}$.