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Further improvement of Jensen inequality and application to stability of time-delayed systems ☆

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ABSTRACT

This paper is concerned with the delay-dependent stability for the linear systems with a time-varying delay. To get a result in the form of LMI from a Lyapunov–Krasovskii functional, an integral inequality is necessary and Jensen inequality has been a most powerful inequality in the last few years. Recently, based on Wirtinger inequality, an improved integral inequality, encompassing Jensen inequality, was proposed and its application to the stability showed a quite improvement. In this paper, without using Wirtinger inequality, a further improved integral inequality in the form of infinite series is derived, and, based on this, a delay-dependent stability condition in the form of LMI is derived. Finally, its contribution on the stability criterion is shown by well-known two examples.

1. Introduction

Time-delay is frequently encountered in many practical systems, and it may lead to the degradation of performance or even instability. Therefore, the stability problem of time-delayed systems has been one of the hot issue in last two decades (Gu, Kharitonov, & Chen, 2003, and see references therein).

Let us consider the time-delayed linear systems

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-d(t)), \\ x(t) = \phi(t), \quad \forall t \in [-h, 0] \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, ϕ is the initial condition, $A, A_1 \in \mathbb{R}^{n\times n}$ are constant matrices, and the time delay satisfies

$$0 \le d(t) \le h, \quad \dot{d}(t) \le \mu. \tag{2}$$

The stability problem is to find a less conservative condition guaranteeing the stability of the system (1) with the constraints (2). To get a delay-dependent result in the form of LMI (Boyd, Ghaoiu, Feron, & Balakrishnan, 1994), the Lyapunov–Krasovskii functional (LKF) has been widely used, where the double integral (or equivalently, weighted single integral) term is essential since its time-derivative contains the size of delay (Fridman & Shaked, 2002; Gu et al., 2003). However, the time-derivative of double integral term contains an integral term which has no equivalent LMI form unfortunately. Therefore, it has been a main issue to derive a less conservative LMI form for the integral term.

The earlier work (Park, 1999) is a pioneer inequality, and it was widely used before the Jensen inequality (Gu et al., 2003) expressed as

$$\begin{aligned} V_{ab}(w) &:= \int_a^b w^T(s)Rw(s)ds \\ &\ge \frac{1}{b-a}\Omega_0^T(w)R\Omega_0(w) := V_{\text{Jensen}}, \end{aligned} \tag{3}$$

where $a < b, R = R^T > 0$ and $\Omega_0(w) = \int_a^b w(s)ds$. The Jensen inequality is a generalized version of Park (1999), and it has been a most powerful tool in the last few years.

Also, the following inequality (Park, Ko, & Jeong, 2011)

$$\begin{bmatrix} \frac{1}{h-\alpha}X_1 & 0 \\ 0 & \frac{1}{\alpha}X_2 \end{bmatrix} \ge \frac{1}{h}\begin{bmatrix} X_1 & S^T \\ S & X_2 \end{bmatrix}, \quad \forall \alpha \in (0, h) \tag{4}$$

makes it possible to get a stability result for the rapid change of time-delay (especially when $\mu \ge 1$), where $X_1 = X_1^T, X_2 = X_2^T$ and $\begin{bmatrix} X_1 & S^T \\ S & X_2 \end{bmatrix} > 0$.

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Recently, based on a Wirtinger inequality, Seuret and Gouaisbaut (2013) presented an analytically improved result that overcomes the Jensen inequality in (3)

$$V_{ab}(w) \geq V_{\text{Jensen}} + \frac{12}{(b-a)^3}\Omega_1^T(w)R\Omega_1(w) := V_{\text{Seuret}} \tag{5}$$

where $\Omega_1(w) = \int_a^b (s - \frac{a+b}{2})w(s)ds$, and its adoption significantly improves the stability criterion (Kwon, Park, Park, Lee, & Cha, 2014; Seuret & Gouaisbaut, 2013).

Similarly, there is also an improvement of the discrete-time version of Jensen inequality (Lam, Zhang, Chen, & Xu, 2015).

2. Further improvement of Jensen inequality

Note that the reduction of gap in the integral inequality is a key to reduce the conservatism for the stability problem. Now, without using a Wirtinger inequality, we give a further improved result in the form of infinite series than the recent integral inequality of Seuret et al. (2013) in (5).

Lemma 1. *For $m = 0, 1, 2, \ldots,$ define*

$$\begin{cases} \psi_{2m}(s) = \left(s - \dfrac{a+b}{2}\right)^{2m} + \displaystyle\sum_{i=0}^{m-1} a_{mi}\left(s - \dfrac{a+b}{2}\right)^{2i} \\ \psi_{2m+1}(s) = \left(s - \dfrac{a+b}{2}\right)^{2m+1} + \displaystyle\sum_{i=0}^{m-1} b_{mi}\left(s - \dfrac{a+b}{2}\right)^{2i+1} \end{cases}$$

with the assumption that, $\forall i = 0, 1, 2, \ldots, m-1$,

$$\int_a^b \psi_{2m}(s)\psi_{2i}(s)ds = \int_a^b \psi_{2m+1}(s)\psi_{2i+1}(s)ds = 0. \tag{6}$$

Then, we have the following integral inequality in the form of infinite series

$$V_{ab}(w) \geq \sum_{i=0}^{\infty} \frac{1}{p_i}\Omega_i^T(w)R\Omega_i(w) \tag{7}$$

where $p_i = \int_a^b \psi_i^2(s)ds > 0$ and $\Omega_i(w) = \int_a^b \psi_i(s)w(s)ds$.

Proof. Note that $\psi_{2i}(s)$ and $\psi_{2i+1}(s)$ are even and odd polynomial w.r.t. $s = \frac{a+b}{2}$, respectively. So, $\int_a^b \psi_{2i}(s)\psi_{2j+1}(s)ds = 0, \forall i, j$. By combining it with the above orthogonality in (6), we have $\int_a^b \psi_i(s)\psi_j(s)ds = 0, \forall i \neq j$.

Now, let $z(s) = \sum_{i=0}^{\infty} \frac{1}{p_i}\psi_i(s)\Omega_i(w)$, then we get

$$\begin{aligned} 0 &\leq \int_a^b [w(s) - z(s)]^T R[w(s) - z(s)]ds \\ &= \int_a^b \left\{ w^T(s)Rw(s) - 2z^T(s)Rw(s) + z^T(s)Rz(s) \right\} ds \\ &= V_{ab}(w) - 2\int_a^b \Big(\sum_{i=0}^{\infty}\frac{1}{p_i}\psi_i(s)\Omega_i(w)\Big)^T Rw(s)ds \\ &\quad + \int_a^b \Big(\sum_{i=0}^{\infty}\frac{1}{p_i}\psi_i(s)\Omega_i(w)\Big)^T R\Big(\sum_{j=0}^{\infty}\frac{1}{p_j}\psi_j(s)\Omega_j(w)\Big)ds \\ &= V_{ab}(w) - 2\sum_{i=0}^{\infty}\frac{1}{p_i}\Omega_i^T(w)R\Big(\int_a^b \psi_i(s)w(s)ds\Big) \\ &\quad + \int_a^b \Big\{\sum_{i=0}^{\infty}\frac{1}{p_i^2}\psi_i^2(s)\Omega_i^T(w)R\Omega_i(w)\Big\}ds \\ &= V_{ab}(w) - 2\sum_{i=0}^{\infty}\frac{1}{p_i}\Omega_i^T(w)R\Omega_i(w) + \sum_{i=0}^{\infty}\frac{1}{p_i^2}p_i\Omega_i^T(w)R\Omega_i(w) \\ &= V_{ab}(w) - \sum_{i=0}^{\infty}\frac{1}{p_i}\Omega_i^T(w)R\Omega_i(w), \end{aligned}$$

which means (7). This completes the proof. □

Remark 1. In Lemma 1, the polynomials $\psi_{2m}(s)$ and $\psi_{2m+1}(s)$ contain the scalars a_{mi} and b_{mi}, $i = 0, 1, 2, \ldots, m-1$, respectively. And these scalars are uniquely determined by (6) since (6) has m algebraic equations for both $\psi_{2m}(s)$ and $\psi_{2m+1}(s)$. Here, some $\psi_i(s)$, $i = 0, 1, 2, 3$, are given:

$$\begin{cases} m = 0: \psi_0(s) = 1, \quad \psi_1(s) = s - \dfrac{a+b}{2}. \\ m = 1: \begin{cases} \psi_2(s) = \left(s - \dfrac{a+b}{2}\right)^2 + a_{10}, \\ \psi_3(s) = \left(s - \dfrac{a+b}{2}\right)^3 + b_{10}\left(s - \dfrac{a+b}{2}\right), \end{cases} \\ \quad \text{where} \begin{cases} \displaystyle\int_a^b \psi_2(s)\psi_0(s)ds = 0 \to a_{10} = -\dfrac{(b-a)^2}{12}, \\ \displaystyle\int_a^b \psi_3(s)\psi_1(s)ds = 0 \to b_{10} = -\dfrac{3(b-a)^2}{20}, \end{cases} \\ \ldots \end{cases}$$

and $p_0 = b - a$, $p_1 = \frac{(b-a)^3}{12}$, $p_2 = \frac{(b-a)^5}{180}$, $p_3 = \frac{(b-a)^7}{2800}, \ldots$. As a result, we have from (7)

$$V_{ab}(w) \geq V_{\text{Seuret}} + \frac{180}{(b-a)^5}\Omega_2^T(w)R\Omega_2(w) + \frac{2800}{(b-a)^7}\Omega_3^T(w)R\Omega_3(w) + \underbrace{\cdots}_{\geq 0}, \tag{8}$$

which shows an analytic improvement compared to the recent result of Seuret et al. (2013) since $R = R^T > 0$.

The following Corollary 1 is a special case of Lemma 1 with $w(s) = \dot{x}(s)$ and $k = 0, 1, 2$, and it will be used in the proof of next main result.

Corollary 1. *Let $a < b$, $R = R^T > 0$, then*

$$-V_{ab}(\dot{x}) \leq -\frac{1}{b-a}\Big\{\Upsilon_0^T(a,b)R\Upsilon_0(a,b) + 3\Upsilon_1^T(a,b)R\Upsilon_1(a,b) + 5\Upsilon_2^T(a,b)R\Upsilon_2(a,b)\Big\} \tag{9}$$

where $\Upsilon_0(a,b) = x(b) - x(a)$, $\Upsilon_1(a,b) = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds$ and $\Upsilon_2(a,b) = x(b) - x(a) - \frac{12}{(b-a)^2}\int_a^b (s - \frac{a+b}{2})x(s)ds$.

Proof. Using $\psi_i(s)$, $i = 0, 1, 2$ in Remark 1, we have

$$\begin{cases} \Omega_0(\dot{x}) = \displaystyle\int_a^b \psi_0(s)\dot{x}(s)ds = \Upsilon(a,b), \\ \Omega_1(\dot{x}) = \displaystyle\int_a^b \psi_1(s)\dot{x}(s)ds = \dfrac{b-a}{2}\Upsilon_1(a,b), \\ \Omega_2(\dot{x}) = \displaystyle\int_a^b \psi_2(s)\dot{x}(s)ds = \dfrac{(b-a)^2}{6}\Upsilon_2(a,b), \end{cases}$$

and it is straightforward to get (9) from (8). So the details are omitted. □

The following Lemma 2 is a result that the quadratic function is negative on a closed interval $[0, h]$ irrespective of its convexity or concavity, and it will be used in the proof of next main result.