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Brief paper

A stopping rule for stochastic approximation☆

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ABSTRACT

A stochastic approximation algorithm is a recursive procedure to find the solution to an unknown nonlinear equation via noisy measurements. In this paper, we present a stopping rule for a stochastic approximation. We show that there is a high probability that the distance between the exact solution and the candidate solution is less than a specified tolerance level when the stochastic approximation stops according to our stopping rule. Furthermore, the number of recursions required by the stopping rule is a polynomial function of the problem size.

1. Introduction

Stochastic approximation (Robbins & Monro, 1951) is a recursive procedure that uses noisy measurements to find the solution to an unknown nonlinear equation. This technique is a part of the basic framework of a probabilistic approach (Tempo, Calafiore, & Dabbene, 2013) to control analysis and synthesis. There are many applications of the control of systems, including recursive identification (Chen, 2010; Mu & Chen, 2013), iterative learning (Butcher, Karimi, & Longchamp, 2008; Chen & Fang, 2004), adaptive control, and consensus control (Huang, Dey, Nair, & Manton, 2010; Huang & Manton, 2010). We can also apply stochastic approximation to unconstrained minimization problems (Kiefer & Wolfowitz, 1952; Spall, 1992; Yousefian, Nedić, & Shanbhag, 2012), since unconstrained optimization is equivalent to finding the zero point of a gradient. Furthermore, the stochastic approximation is one of the key ideas of distributed optimization (Nedić & Ozdaglar, 2009). That is, it is important to investigate the method.

There are much research on the theory of stochastic approximation, and sufficient conditions are now well established for the convergence of the candidate solutions that are generated by this procedure (Benveniste, Métivier and Priouret, 1990; Borkar, 2008; Chen, 2002; Kushner & Yin, 1997; Nevelson & Hasminskii, 1976; Spall, 2003). In particular, it is established that stochastic approximation yields the solution of the unknown equation if the procedure is iterated sufficiently many (possibly infinitely many) times.

However, in practical situations, the algorithm must be stopped after a finite number of recursions, and we thus focus on the quality of the candidate solutions obtained from a finite number of samples. Although Polyak (1976) employs Lyapunov function and Martingale theory to clarify the rate of convergence, they do not determine when the procedure should stop. We therefore wish to develop a stopping rule that would guarantee a sufficiently small distance between a candidate solution and the exact solution.

There are little research on determining stopping rules for stochastic approximation. Burkholder (1956) shows that the candidate solutions for the scalar stochastic approximation asymptotically follow a normal distribution; that is, they have *asymptotic normality*. Based on this asymptotic normality, some stopping rules (Sielken, 1973; Stroup & Braun, 1982) are developed for the stochastic approximation of scalar nonlinear equations. Stopping rules are shown in Yin (1988) for the simple multidimensional case and in Yin (1990) for the multidimensional case in which the measurement noise depends on the candidate solutions. Notice that these stopping rules provide *practical* guarantees on the size of the estimation error, which is the distance between the exact solution of the unknown equation and the current candidate solution. Since this property is only guaranteed for a large number of recursions for these stopping rules, the necessary number of recursions is unknown in advance. In fact, it is shown that the obtained solutions are generally not close enough to the exact solution when the number of recursions is small (Glynn & Whitt, 1992). As another

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research on stopping rules, Pflug (1990) proposes a stopping rule for stochastic approximation that has a constant step size. The rule guarantees that the obtained solution is close enough to the exact solution of the unknown equation. However, it requires a posteriori information, in particular, it is necessary to know the signs of the measurements used in the procedure, and thus we cannot know the necessary number of recursions in advance.

In this paper, we present a stopping rule that is based only on information that is available a priori. We have already provided a stopping rule for linear stochastic approximation (Wada, Itani, & Fujisaki, 2010). However, since stochastic approximation is essentially a method for unknown "nonlinear" equation, we have to develop stopping rules for nonlinear stochastic approximation. For achieving this goal, we will consider basic stochastic approximation for a nonlinear equation. We then derive an explicit bound on the number of recursions required in order to achieve a given small average estimation error. This average estimation error is specified by two parameters. The first is related to the initial distance between the candidate solution and the exact solution of the equation. The second is related to the effect of the observation noise. The required number of recursions is bounded by a polynomial function of the covariance of the noise, the parameters that specify the average estimation error. The key idea used for constructing the proposed stopping rule is the fact that the candidate solution behaves like a time-varying system that is driven by stochastic noise.

This paper is organized as follows. In Section 2, we present a stopping rule for the standard stochastic approximation. In Section 3, we present some numerical examples. Next, we provide the proof of the stopping rule discussed in Section 4, and we present our concluding remarks in Section 5.

2. A stopping rule for stochastic approximation

Let us consider a nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is unknown, but its measurement

$$y = f(x) + \xi \tag{2}$$

at a given $x \in \mathbb{R}^n$ is available, although it contains measurement noise $\xi \in \mathbb{R}^n$. We assume that the solution x^* of the nonlinear equation (1) is unique. Furthermore, we make the following assumptions about the function f and the measurement noise.

Assumption 1. The function $f(\cdot)$ is differentiable at any $x \in \mathbb{R}^n$. In addition to this, there exist $\zeta \in (0, \infty)$ and $\eta \in (0, 1)$ such that the gradient $\nabla f(x)$ of f with respect to x satisfies

$$\left\| \int_0^1 I - \frac{\nabla f(x^* + t(x - x^*))}{\zeta} \mathrm{d}t \right\| \le 1 - \eta \tag{3}$$

for any $x \neq x^*$, where I is the identity matrix and $\|\cdot\|$ denotes the matrix norm induced by the Euclidean norm.

Notice that there exist ζ and η such that the above assumption holds if a nonlinear continuous time system $\dot{x} = -f(x)$ is quadratic stable and $\|\nabla f(x)\|$ is bounded, as shown below in Lemma 9. Then, the assumption implies the uniqueness of the solution to problem (1). A type of quadratic stability is one of the standard assumptions used for stopping rules; for example, see Yin (1988) and Yin (1990). When we analyze convergence property only, differentiability of f is not needed (Chen, 2002). However, construction of a stopping rule requires detailed analysis on transient behavior of the candidate solution. Furthermore, our stopping rule does not employ any measurement information. Thus, to develop a stopping rule, we need differentiation of equation f. Furthermore, even if we do not know any information on the function f, we can estimate necessary number of recursions by setting sufficiently small η and sufficiently large ζ.

Assumption 2. The measurement noise $\xi \in \mathbb{R}^n$ is a random vector. Then, its mean is zero and its variance $\Sigma \ge 0$ is bounded, i.e.,

$$\mathsf{E}[\xi] = 0,$$
$$\mathsf{Var}[\xi] = \mathsf{E}\left[(\xi - \mathsf{E}[\xi])(\xi - \mathsf{E}[\xi])^{\mathrm{T}}\right] = \Sigma \ge 0,$$

where $\mathsf{E}[\cdot]$ denotes the expectation and $\mathsf{Var}[\cdot]$ is its variance.

Standard stochastic approximation (Robbins & Monro, 1951) is a recursive procedure for solving the nonlinear equation (1) according to the update rule

$$x_{k+1} = x_k - a_k y_k, \tag{4}$$

where k denotes the recursion number. That is, $x_k \in \mathbb{R}^n$ is the kth candidate for the solution, $y_k \in \mathbb{R}^n$ is the kth measurement of Eq. (1) according to (2), and a_k is the kth step size, as specified by the user. Under appropriate assumptions, the stochastic approximation asymptotically gives us the solution x^* of (1). For example, some sufficient conditions for convergence have been summarized in Benveniste et al. (1990), Borkar (2008), Kushner and Yin (1997), Nevelson and Hasminskii (1976), Spall (2003). In this paper, we assume that $\{\xi_i, i = 1, 2, \dots, \}$ is an independent random sequence and we select

$$a_k = \frac{1}{\eta\zeta(k_0 + k)},$$

where k_0 is a stability constant, and introduce the following assumptions for ensuring convergence of the procedure (4).

Under these conditions, we have the following stopping rule.

Theorem 3. *When Assumptions 1 and 2 are satisfied, for given constants $\alpha \in (0, \infty)$ and $\beta \in (0, \infty)$, select $\bar{k} \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ that satisfy*

$$\bar{k} \ge \max\{\tau_1, \tau_2\}, \tag{5}$$

$$\tau_1 := \frac{k_0 + 1}{\sqrt{\alpha}} - k_0, \tag{6}$$

$$\tau_2 := \frac{4\mathrm{Tr}\,\Sigma}{\beta\eta^2\zeta^2} - k_0, \tag{7}$$

$$k_0 \ge \frac{1}{\eta} - 1. \tag{8}$$

Then, for any initial candidate $x_1 \in \mathbb{R}^n$ for the solution, the $\bar{k}$th solution candidate $x_{\bar{k}}$ generated by (4) satisfies

$$\mathsf{E}[\|x_{\bar{k}} - x^*\|^2] \le \alpha\|x_1 - x^*\|^2 + \beta, \tag{9}$$

where $\|\cdot\|$ is the Euclidean norm.

Since the proof of this theorem is somewhat complicated, we will wait to present it until Section 4. This theorem states that, if we select an appropriate number $\bar{k}$ of recursions, then $x_{\bar{k}}$ is close to the solution x^* of Eq. (1) in the second mean, that is, the average distance between $x_{\bar{k}}$ and x^* is determined by α and β, which are freely selected by the user. Furthermore, τ_1 and τ_2 are polynomial functions of $1/\alpha$, $1/\beta$, $1/\eta$, $1/\zeta$, and $\mathrm{Tr}\,\Sigma$. Such an appropriate $\bar{k}$ clarifies the probabilistic relationship between the properties of the noise and the estimation error. Furthermore, to determine $\bar{k}$, it is not necessary to have a posteriori information, such as the measurement sequence $y_k, k = 1, 2, \dots$. This is very different from the information needed in conventional stopping rules (Sielken, 1973; Stroup & Braun, 1982; Yin, 1988, 1990). (See Appendix B.)

Our result is a natural extension of conventional results. When we solve (6) and (7) with respect to α and β, we have

$$\alpha = \left(\frac{k_0 + 1}{k + k_0}\right)^2, \qquad \beta = \frac{4\mathrm{Tr}\,\Sigma}{\eta^2\zeta^2(k + k_0)}.$$