Automatica 60 (2015) 182–188

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# On the sample size of random convex programs with structured dependence on the uncertainty ☆

Xiaojing Zhang [a], Sergio Grammatico [b], Georg Schildbach [c], Paul Goulart [d], John Lygeros [a]

[a] Automatic Control Laboratory, ETH Zurich, Switzerland
[b] Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands
[c] Model Predictive Control Lab, UC Berkeley, United States
[d] Department of Engineering Science, Oxford University, United Kingdom

ARTICLE INFO

*Article history:*
Received 14 July 2014
Received in revised form
25 May 2015
Accepted 29 June 2015
Available online 30 July 2015

*Keywords:*
Scenario optimization
Random convex programs
Chance constraints
Randomized Model Predictive Control
Stochastic Model Predictive Control

ABSTRACT

The "scenario approach" provides an intuitive method to address chance constrained problems arising in control design for uncertain systems. It addresses these problems by replacing the chance constraint with a finite number of sampled constraints (scenarios). The sample size critically depends on Helly's dimension, a quantity always upper bounded by the number of decision variables. However, this standard bound can lead to computationally expensive programs whose solutions are conservative in terms of cost and violation probability. We derive improved bounds of Helly's dimension for problems where the chance constraint has certain structural properties. The improved bounds lower the number of scenarios required for these problems, leading both to improved objective value and reduced computational complexity. Our results are generally applicable to Randomized Model Predictive Control of chance constrained linear systems with additive uncertainty and affine disturbance feedback.

© 2015 Elsevier Ltd. All rights reserved.

## 1. Introduction

Many problems in systems analysis and control synthesis can be formulated as optimization problems, including Lyapunov stability, robust control, and Model Predictive Control (MPC) problems (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Mayne, 2014; Tempo, Calafiore, & Dabbene, 2013). In reality, most systems are affected by uncertainty and/or disturbances, in which case a control decision should be made that accounts for these uncertainties. In robust optimization, one seeks a solution satisfying all admissible uncertainty realizations (worst-case approach). Unfortunately, robust programs are in general difficult to solve (Ben-Tal & Nemirovski, 1998), and computational tractability is often obtained at the cost of introducing conservatism.

Stochastic optimization offers an alternative approach where constraints are interpreted in a probabilistic sense as chance constraints, allowing for constraint violations with limited probability (Prékopa, 1995). Except for special cases, chance constrained problems are also intractable, since they generally are non-convex and require the computation of high-dimensional probability integrals. Randomized methods are tools for approximating the solutions to such problems, without being limited to specific probability distributions. By replacing the chance constraint with a finite number of randomly sampled constraints, the fundamental question in randomized algorithms is how large to choose the sample size to guarantee constraint satisfaction with high confidence. One approach is based on the Vapnik–Chervonenkis (VC) theory of statistical learning (Anthony & Biggs, 1992), which has been studied widely for control applications (Tempo et al., 2013; Vidyasagar, 2001). Recently, a new randomized method known as the "scenario approach" has emerged, which is applicable whenever the sampled program is convex (Calafiore, 2010; Campi & Garatti, 2008), and which has been successfully exploited for control design (Calafiore & Campi, 2006). Compared to methods based on the theory of statistical learning, the sample size required by the scenario approach is typically much lower (Calafiore & Campi, 2005, Section 1.2).

The sample size bounds provided by the scenario approach are based on the notion of *Helly's dimension* (Calafiore, 2010, Definition

☆ This research was partially funded by the Swiss Nano-Tera.ch under the project HeatReserves and the European Commission under project SPEEDD (FP7-ICT 619435). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Martin Guay under the direction of Editor Ian R. Petersen.

*E-mail addresses:* xiaozhan@control.ee.ethz.ch (X. Zhang), grammatico@control.ee.ethz.ch, s.grammatico@tue.nl (S. Grammatico), schildbach@berkeley.edu (G. Schildbach), paul.goulart@eng.ox.ac.uk (P. Goulart), jlygeros@control.ee.ethz.ch (J. Lygeros).

http://dx.doi.org/10.1016/j.automatica.2015.07.013
0005-1098/© 2015 Elsevier Ltd. All rights reserved.

3.3), which is always upper bounded by the number of decision variables (Calafiore, 2010, Lemma 2.2). Since the sample size bound grows linearly in Helly's dimension (Calafiore, 2010, Corollary 5.1), finding better bounds not only reduces conservatism of the solution, but also allows problems to be solved faster. Unfortunately, computing Helly's dimension for a given problem is often challenging. To the best of the authors' knowledge, the only attempts to obtain improved bounds are the works in Schildbach, Fagiano, and Morari (2013) and Zhang, Grammatico, Schildbach, Goulart, and Lygeros (2014). It is shown in Schildbach et al. (2013) that Helly's dimension can be upper bounded by the so-called *support rank* (s-rank), obtained by exploiting structural properties of the constraints in the *decision space*. Under certain technical assumptions, the authors of Zhang et al. (2014) upper bound Helly's dimension by the number of active constraints, which can be applied for cases where the constraint functions are affine in the uncertainty variables.

In this paper, we propose new methodologies for bounding Helly's dimension that exploit additional structure in the constraint functions. We first establish bounds for generic problems where the constraint functions are separable in the decision and uncertainty variables. We then exploit these structures for cases where the constraint functions depend affinely and quadratically on the uncertainty variables. The derived sample size depends on the dimension of the *uncertainty space*, hence complementing (Schildbach et al., 2013) and generalizing those in Zhang et al. (2014). Furthermore, we also show explicitly that for the considered problems the scenario approach together with our bounds always provides lower sample sizes than the corresponding ones based on the VC theory of statistical learning.

## 2. Problem description and technical background

Let $\delta \in \Delta \subseteq \mathbb{R}^d$ be a random variable defined on a probability space $(\Delta, \mathcal{F}, \mathbb{P})$. We consider chance constrained problems (CCPs) of the form

$$\text{CCP}(\epsilon): \begin{cases} \min_{x\in\mathcal{X}} & c^\top x \\ \text{s.t.} & \mathbb{P}\left[g(x,\delta) \le 0\right] \ge 1-\epsilon, \end{cases} \tag{1}$$

where $\mathcal{X} \subset \mathbb{R}^n$ is a compact convex set, $x \in \mathbb{R}^n$ the decision variable, $g : \mathbb{R}^n \times \Delta \to \mathbb{R}$ the constraint function, $\epsilon \in (0, 1)$ the acceptable violation probability, and $c \in \mathbb{R}^n$ the cost vector. We consider the scenario program (SP) associated with CCP($\epsilon$), where the chance constraint in (1) is replaced by $N$ sampled constraints, corresponding to independent identically distributed (i.i.d.) realizations $\delta^{(1)}, \dots, \delta^{(N)} \in \Delta$ of the uncertainty vector $\delta$ (Calafiore & Campi, 2005, 2006):

$$\text{SP}[\omega]: \begin{cases} \min_{x\in\mathcal{X}} & c^\top x \\ \text{s.t.} & g(x,\delta^{(j)}) \le 0 \quad \forall j \in \{1,\dots,N\}. \end{cases} \tag{2}$$

We refer to $\omega := \{\delta^{(1)}, \dots, \delta^{(N)}\} \in \Delta^N$ as a multi-sample. Throughout this paper, we make the following assumption.

**Standing Assumption 1** (*Regularity*). For almost all $\delta \in \Delta$, the function $x \mapsto g(x,\delta)$ is *convex* and *lower semi-continuous*. For any integer $N$, SP[$\omega$] in (2) is almost surely *feasible*; its optimizer *exists* and is *unique* for almost all realizations of $\omega \in \Delta^N$. For all $x \in \mathcal{X}$, the mapping $\delta \mapsto g(x,\delta)$ is measurable.

Standing Assumption 1 is standard in the scenario approach (Campi & Garatti, 2008, Assumption 1), (Calafiore, 2010, Assumptions 1, 2), (Calafiore & Campi, 2006, Assumptions 1, 2), and (Grammatico, Zhang, Margellos, Goulart, & Lygeros, 2016, Appendix B). The uniqueness requirement can be relaxed by adopting a suitable (strictly convex or lexicographic) tie-break rule (Calafiore & Campi, 2005, Section 4.1).

Let us denote the (unique) minimizers of SP[$\omega$] and SP[$\omega \setminus \{\delta^{(k)}\}$], for $k \in \{1,\dots,N\}$, by $x^\star$ and $x_k^\star$, respectively. Our forthcoming results are based on the following two key definitions.

**Definition 1** (*Support Constraint Calafiore & Campi, 2005, Definition 4*). The sample $\delta^{(k)}$ is called a *support sample* if $c^\top x_k^\star < c^\top x^\star$; in this case the corresponding constraint $g(\cdot, \delta^{(k)})$ is called a *support constraint* for SP[$\omega$]. The *set of support constraints* of SP[$\omega$] is denoted by sc(SP[$\omega$]).

We denote by $|\text{sc}(\text{SP}[\omega])|$ the cardinality of the set of support constraints.

**Definition 2** (*Helly's Dimension Calafiore, 2010, Definition 3.1*). *Helly's dimension* of SP[$\omega$] in (2) is the smallest integer $\zeta$ such that $\operatorname{ess\,sup}_{\omega\in\Delta^N} |\text{sc}(\text{SP}[\omega])| \le \zeta$ holds for any finite $N \ge 1$.

Intuitively, the SP in (2) can be used to approximate the CCP in (1). Indeed, the authors of Campi and Garatti (2008) and Calafiore (2010) show that if the sample size $N$ satisfies

$$\sum_{j=0}^{\zeta-1} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j} \le \beta \tag{3}$$

for some $\beta \in (0, 1)$, then, with confidence at least $1-\beta$, the optimal solution of SP[$\omega$] is feasible for the original CCP($\epsilon$) (Calafiore, 2010, Theorem 3.3). It was shown in Calafiore (2010, Lemma 2.2) that Helly's dimension $\zeta$ is always upper bounded by $n$. This *standard bound* ($\zeta \le n$), however, is only tight for fully-supported problems (Campi & Garatti, 2008, Theorem 1), but remains conservative otherwise.

The overall goal of this paper is to find tighter upper bounds on Helly's dimension $\zeta$ for non-fully-supported problems, which would allow for smaller $N$ than the one given by the standard bound. Following Calafiore (2010) one can show that (3) is satisfied if $N$ is chosen such that

$$N \ge \frac{2}{\epsilon}\left(\zeta - 1 + \ln\left(\frac{1}{\beta}\right)\right). \tag{4}$$

Since $\epsilon$ is typically chosen small in many practical applications (e.g. $10^{-1}$–$10^{-4}$) and $N$ roughly scales as $\mathcal{O}(\zeta/\epsilon)$, finding a good bound on $\zeta$ is key for reducing the required sample size. A small sample size is attractive mainly for two reasons: less conservative solutions in terms of cost, and reduced computational time for solving the scenario program in (2).

### 2.1. Bounding Helly's dimension

Unfortunately, explicitly computing $\zeta$ is in general very difficult. Tighter bounds on Helly's dimension were introduced in Schildbach et al. (2013), based on the so-called *support rank* (s-rank). It is defined as the dimension $n$ of the decision space minus the dimension of the maximal linearly unconstrained subspace (Schildbach et al., 2013, Definition 3.6), and is therefore never worse than the standard bound.

There are, however, cases where the s-rank yields no improvement upon the standard bound, although the exact Helly's dimension is much lower. Consider, for instance,

$$\begin{cases} \min_{(y,h)\in\mathbb{R}^{n-1}\times\mathbb{R}} & h \\ \text{s.t.} & \mathbb{P}\left[\|Ay-b\| + \delta \le h\right] \ge 1-\epsilon, \end{cases} \tag{5}$$

where $\|\cdot\|$ is any norm, $\delta \in \mathbb{R}$ is a continuous random variable, $A \in \mathbb{R}^{k\times(n-1)}$ has full column rank and $b \in \mathbb{R}^k$. The s-rank for the above problem is $n$, because $A$ is full column rank. Hence,

Download English Version:

https://daneshyari.com/en/article/695361

Download Persian Version:

https://daneshyari.com/article/695361

Daneshyari.com