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Observability conservation by output feedback and observability Gramian bounds☆

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ABSTRACT

Though it is a trivial fact that the observability of a linear state space system is conserved by output feedback, it requires a rigorous proof to generalize this result to *uniform complete observability*, which is defined with the observability Gramian. The purpose of this paper is to present such a proof. Some issues in existing results are also discussed. The uniform complete observability of closed loop systems is useful for the analysis of some adaptive systems and of the Kalman filter.

1. Introduction

For a linear time varying (LTV) state space system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1a}$$

$$y(t) = C(t)x(t) \tag{1b}$$

with the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}^k$, the output $y(t) \in \mathbb{R}^m$, and with bounded piecewise continuous matrices $A(t), B(t), C(t)$ of appropriate sizes, it is well known that its observability depends only on the matrix pair $[A(t), C(t)]$.

Consider the output feedback $u(t) = -L(t)y(t)$ with some matrix $L(t) \in \mathbb{R}^{k\times m}$ and let $K(t) \triangleq B(t)L(t) \in \mathbb{R}^{n\times m}$, then the closed loop system

$$\dot{x}(t) = A(t)x(t) - K(t)y(t) \tag{2a}$$

$$y(t) = C(t)x(t) \tag{2b}$$

and the equivalent system

$$\dot{x}(t) = (A(t) - K(t)C(t))x(t) \tag{3a}$$

$$y(t) = C(t)x(t) \tag{3b}$$

have the same observability property as (1).

It is then clear that, if the matrix pair $[A(t), C(t)]$ is observable, then so is the matrix pair $[(A(t) - K(t)C(t)), C(t)]$. The converse is also true. In Anderson et al. (1986, page 38), the statement of this result was generalized to *uniform complete observability* (UCO, definition recalled in the next section), but without being proved.

Proofs of the generalized result can be found in Sastry and Bodson (1989) and Ioannou and Sun (1996), but some details of these proofs merit revision and comments, as discussed in the Appendix. The purpose of this paper is to propose a complete proof with new observability Gramian bounds.

In Aeyels, Sepulchre, and Peuteman (1998) a similar result is presented, but it does not provide an estimation of the bounds of the observability Gramian, and assumes an extra condition: the uniform stability of the considered systems, which is not required in the present paper.

Matrices of the form $(A(t) - K(t)C(t))$ appear naturally in the error dynamics equation of the Kalman filter, and similarly in state observers. The UCO of the matrix pair $[(A(t) - K(t)C(t)), C(t)]$ helps to establish the stability of the Kalman filter or of state observers (Kalman, 1963). It is also useful for the analysis of some adaptive systems (Anderson et al., 1986, chapter 2), (Sastry & Bodson, 1989, chapter 2), (Ioannou & Sun, 1996, chapter 4).

2. Observability Gramian bounds

Let $\Phi(t, t_0)$ denote the state transition matrix of system (1). The observability Gramian of this system is

$$M(t_1, t_2) = \int_{t_1}^{t_2} \Phi^T(t, t_2)C^T(t)C(t)\Phi(t, t_2)\mathrm{d}t. \tag{4}$$

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System (1) is said *uniformly completely observable* (UCO[1]) (Kalman, 1963, p. 358) if there exist positive constants σ, α, β such that the inequalities (I_n denotes the $n \times n$ identity matrix)

$$0 < \alpha I_n \le M(t-\sigma, t) \le \beta I_n \quad (5)$$

hold for all $t \in \mathbb{R}$. Similarly, let $\tilde{\Phi}(t, t_0)$ be the state transition matrix of system (3), and its observability Gramian

$$\tilde{M}(t_1, t_2) = \int_{t_1}^{t_2} \tilde{\Phi}^T(t, t_2) C^T(t) C(t) \tilde{\Phi}(t, t_2) \mathrm{d}t. \quad (6)$$

System (3) is said UCO if there exist positive constants $\tilde{\sigma}$, $\tilde{\alpha}$, $\tilde{\beta}$ such that, for all $t \in \mathbb{R}$,

$$0 < \tilde{\alpha} I_n \le \tilde{M}(t-\tilde{\sigma}, t) \le \tilde{\beta} I_n. \quad (7)$$

The following result was stated in Anderson et al. (1986, p. 38, Lemma 2.3).

Lemma 1. *The matrix pair $[A(t), C(t)]$ is UCO, if and only if for any bounded and locally integrable matrix $K(t)$ the matrix pair $[(A(t) - K(t)C(t)), C(t)]$ is UCO.* □

For a rigorous proof of this result, it should be shown that the existence of σ, α, β implies the existence of $\tilde{\sigma}$, $\tilde{\alpha}$, $\tilde{\beta}$, and vice versa. No such proof was given in Anderson et al. (1986). The same lemma (up to minor differences) is presented in Sastry and Bodson (1989) and Ioannou and Sun (1996) with similar proofs, but some details merit revision and comments, as discussed in the Appendix.

Throughout this paper, "$\|\cdot\|$" will denote the Euclidean vector norm or the matrix norm induced by the Euclidean vector norm.

Let us prove the following more complete result.

Lemma 2. *Assume that the positive constants γ, η, ρ are such that the inequalities*

$$\|A(t)\| \le \gamma, \qquad \|C(t)\| \le \eta, \qquad \|K(t)\| \le \rho \quad (8)$$

hold for all $t \in \mathbb{R}$, and that the observability Gramian of the matrix pair $[A(t), C(t)]$ satisfies the inequalities (5) with some positive constants σ, α, β, then the observability Gramian of the matrix pair $[(A(t) - K(t)C(t)), C(t)]$ satisfies the inequalities (7) with $\tilde{\sigma} = \sigma$ and

$$\tilde{\alpha} = \begin{cases} \left(\dfrac{\sqrt{\beta - \alpha + \varphi\alpha} - \sqrt{\beta}}{\varphi - 1} \right)^2 > 0 & \text{if } \varphi \neq 1 \\ \dfrac{\alpha^2}{4\beta} > 0 & \text{if } \varphi = 1 \end{cases} \quad (9a)$$

$$\tilde{\beta} = \left(\sqrt{\beta - \alpha + \psi\beta} + \sqrt{\beta} \right)^2 \quad (9b)$$

where

$$\varphi = \eta^2\rho^2 \left(\frac{e^{2\gamma\sigma} - 1}{4\gamma^2} - \frac{\sigma}{2\gamma} \right) \quad (10)$$

$$\psi = \eta^2\rho^2 \left(\frac{e^{2(\gamma+\eta\rho)\sigma} - 1}{4(\gamma+\eta\rho)^2} - \frac{\sigma}{2(\gamma+\eta\rho)} \right). \quad □ \quad (11)$$

Notice the "symmetry" between $A(t)$ and

$$\tilde{A}(t) \triangleq A(t) - K(t)C(t), \quad (12)$$

in the sense that

$$A(t) = \tilde{A}(t) - (-K(t))C(t), \quad (13)$$

hence the converse of Lemma 2 is implied by the lemma itself.

[1] In this paper "UCO" is used either as a noun or as an adjective. Some variants of the definition exist (Ioannou & Sun, 1996; Sastry & Bodson, 1989). The definition recalled here follows Kalman (1963).

The proof of Lemma 2 will need the following result.

Lemma 3. *Let $\Phi(t, t_0)$ be the state transition matrix of $\dot{x}(t) = A(t)x(t)$ with $\|A(t)\| \le \gamma$, then $\|\Phi(t, t_0)\| \le e^{\gamma|t-t_0|}$ for all t, $t_0 \in \mathbb{R}$.* □

See Chicone (2006), Theorem 2.4. for a proof of this result.

Proof of Lemma 2. Let us first establish the relationship between $\Phi(t, t_0)$ and $\tilde{\Phi}(t, t_0)$.

The solution of (3a), with some initial state $x(t_0)$, is

$$x(t) = \tilde{\Phi}(t, t_0)x(t_0). \quad (14)$$

On the other hand, rewrite (3a) as

$$\dot{x}(t) = A(t)x(t) - K(t)C(t)x(t) \quad (15)$$

and treat $-K(t)C(t)x(t) = -K(t)C(t)\tilde{\Phi}(t, t_0)x(t_0)$ as the input term, then

$$x(t) = \Phi(t, t_0)x(t_0) - \int_{t_0}^{t} \Phi(t, p)K(p)C(p)\tilde{\Phi}(p, t_0)x(t_0)\mathrm{d}p. \quad (16)$$

For *any* initial state $x(t_0) \in \mathbb{R}^n$, the right hand sides of (14) and of (16) are equal, hence

$$\tilde{\Phi}(t, t_0) = \Phi(t, t_0) - \int_{t_0}^{t} \Phi(t, p)K(p)C(p)\tilde{\Phi}(p, t_0)\mathrm{d}p.$$

This equation holds for *any* pair of real numbers (t, t_0), which can be replaced by any other pairs, say (s, t),

$$\tilde{\Phi}(s, t) - \Phi(s, t) = -\int_{t}^{s} \Phi(s, p)K(p)C(p)\tilde{\Phi}(p, t)\mathrm{d}p. \quad (17)$$

Left multiply both sides by $C(s)$ and right multiply by any (arbitrary) unit vector $v \in \mathbb{R}^n$, and take the integral of the squared norm of both sides, then

$$\int_{t-\sigma}^{t} \left\| C(s) \left(\tilde{\Phi}(s, t) - \Phi(s, t) \right) v \right\|^2 \mathrm{d}s$$
$$= \int_{t-\sigma}^{t} \left\| C(s) \int_{s}^{t} \Phi(s, p)K(p)C(p)\tilde{\Phi}(p, t)v\mathrm{d}p \right\|^2 \mathrm{d}s.$$

Develop the squared Euclidean norm at the left hand side and rearrange the terms, then

$$\int_{t-\sigma}^{t} \|C(s)\Phi(s, t)v\|^2 \mathrm{d}s = -\int_{t-\sigma}^{t} \left\| C(s)\tilde{\Phi}(s, t)v \right\|^2 \mathrm{d}s$$
$$+ \int_{t-\sigma}^{t} 2v^T\Phi^T(s, t)C^T(s)C(s)\tilde{\Phi}(s, t)v\mathrm{d}s$$
$$+ \int_{t-\sigma}^{t} \left\| C(s) \int_{s}^{t} \Phi(s, p)K(p)C(p)\tilde{\Phi}(p, t)v\mathrm{d}p \right\|^2 \mathrm{d}s. \quad (18)$$

Each of the terms involved in (18) will be examined in the following in order to derive an inequality bounding the observability Gramian.

As $\|v\| = 1$ (unit vector), it follows from (5) that

$$\alpha = v^T\alpha I_n v \le v^T M(t-\sigma, t)v$$
$$= \int_{t-\sigma}^{t} \|C(s)\Phi(s, t)v\|^2 \mathrm{d}s \le v^T\beta I_n v = \beta,$$

therefore

$$\alpha \le \int_{t-\sigma}^{t} \|C(s)\Phi(s, t)v\|^2 \mathrm{d}s \le \beta. \quad (19)$$

Define

$$\chi \triangleq \int_{t-\sigma}^{t} \left\| C(s)\tilde{\Phi}(s, t)v \right\|^2 \mathrm{d}s = v^T\tilde{M}(t-\sigma, t)v. \quad (20)$$