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Vibrational control without averaging☆

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ABSTRACT

The standard approach to vibrational control applies first-order averaging methods to find an open-loop periodic input that stabilizes an unstable equilibrium point. While the capability of stabilization without feedback is appealing, this formulation has drawbacks from a design perspective. An alternative design framework based on stability maps for second-order linear periodic systems is not as general, but has significant potential advantages. The averaging approach only guarantees that a solution exists; the designer must then find that solution by other means. Furthermore, the frequencies required may be too high for practical implementation. Use of stability maps makes a broader class of stabilizing inputs accessible, allowing, for example, the use of lower frequency signals. Application to nonlinear and higher-order systems is demonstrated with two examples. The first is stabilization of the classical vertically forced inverted pendulum. The second is delay of a pitchfork bifurcation in a fourth-order nonlinear system. In the second example we show that the averaging-based approach necessarily fails to delay the bifurcation, while the alternative method achieves significant extension of the stable operating region.

1. Introduction

Vibrational control is a method for stabilizing an unstable equilibrium point using periodic open-loop inputs of appropriate frequency and amplitude. The possibility of stabilization without feedback is especially appealing for systems with limited sensing and/or actuation, or with many degrees of freedom. The use of an open-loop periodic input as an explicit control strategy was first presented by Meerkov in 1980 for linear time-varying systems, with applications to control of distributed chemical processes (Meerkov, 1980). This influential paper cited results from averaging theory to support its main theorems. A series of papers by Bentsman, Bellman, and Meerkov further extended the use of averaging methods (Bellman, Bentsman, & Meerkov, 1985, 1986; Bentsman, 1987; Meerkov, 1982). Baillieul and co-workers adapted averaging methods to a geometric framework, and extended vibrational control to conservative systems (Baillieul, 2008; Weibul & Baillieul, 1997, 1998; Weibul, Kaper, & Baillieul, 1997). Nonaka and Baillieul and co-workers applied averaging-based vibrational control to electromagnetic and electrostatic actuators, including applications to microsystems (Nonaka, Baillieul, & Horenstein, 2001; Nonaka, Sakai, & Baillieul, 2004; Nonaka, Sugimoto, & Baillieul, 2004; Sugimoto, Nonaka, & Horenstein, 2005).

Averaging methods address approximation of solutions of a time-varying system by solutions to an averaged autonomous system (Sanders, Verhulst, & Murdock, 2007). Averaging methods continue to be an active area of research for oscillatory control of Lagrangian and Hamiltonian systems, along with more general nonlinear and geometric formulations (Bombrun & Pomet, 2013; Bullo, 2002; Dimeo & Thompoulos, 1994; Hong, Lee, & Lee, 1998; Martinez, Cortes, & Bullo, 2003; Sanyal, Bloch, & Harris McClamroch, 2005; Tahmasian, Taha, & Woolsey, 2013; Teel, Peuteman, & Aeyels, 1999; Vela & Burdick, 2003a,b,c). Further extensions include application of higher-order averaging and series expansion methods for nonlinear systems. Strong connections have been made between averaging theory and linear and nonlinear Floquet theory (Vela, 2003). These topics are beyond the scope of the present paper, which is intended to provide insight into limitations of the standard averaging approach to vibrational control, and to suggest an alternative design framework that may bypass these limitations. Subsequently this paper considers only

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first-order averaging methods to determine the stability of an equilibrium point.

Broadly speaking, the first step in the standard approach to vibrational control is to show that the stability of an equilibrium point at the origin of a time-varying system $\dot{x} = \epsilon f(t/\epsilon, x)$ is equivalent to stability of an equilibrium at the origin of a time-invariant *averaged* system $\dot{y} = F(y)$ when the parameter ϵ is sufficiently small. Formally, this involves showing the existence of a threshold value $\epsilon_\star$ such that the system is stable for all $\epsilon \in [0, \epsilon_\star]$. A control term is often introduced with the form $(1/\epsilon)\phi(t/\epsilon, x)$, where ϕ is periodic in its first argument. In this context the parameter ϵ may be replaced by $1/\omega$ where ω is interpreted as the frequency of the forcing, and stability properties are obtained for $\omega \in [\omega_\star, \infty)$. Averaging methods provide sufficient conditions for the *existence* of the threshold value, however the theorems typically used do not provide a *value* for $\omega_\star$. To use the averaging results for control, one first stabilizes the averaged system and then explores increasing ω by other means – such as simulation – until the original time-varying system becomes stable.

While averaging provides a powerful analytical framework, the lack of an explicit value for the threshold frequency is inconvenient for design, and the resulting stabilizing input frequency may be impractically high for implementation. Furthermore, as is shown subsequently, the threshold frequency will itself vary depending on the specific way in which the control term is initially specified, and the standard approach offers no guidance for making an advantageous choice. It is also shown below that the first stable response in an increasing frequency sweep may not correspond to $\omega_\star$, potentially misleading a designer and resulting in instability for a higher frequency input. On the other hand, the results presented subsequently also show that inputs that do not satisfy the condition for averaged stability may be acceptable – even preferable – for control. Such inputs are not accessible from the averaging framework. In short, the examples show that the condition for average stability is neither necessary nor sufficient for stability of the original system at a particular input frequency, nor does the condition provide explicit guidance for the choice of a stabilizing frequency. These examples suggest that there are potential benefits to applying methods of vibrational control that do not directly rely on averaging theory.

Stabilization using open-loop oscillatory excitation has a long history outside the control community, and while averaging methods are often used, other techniques have also proven effective. Perhaps the oldest and best known example of stabilization by oscillatory excitation is the inverted simple pendulum forced by periodic vertical base motion. In 1908, A. Stephenson used periodic series solutions to analyze stability of a linearized version of this system. That result was restricted to small base motions and high forcing frequencies (Stephenson, 1908). In 1951, P.L. Kapitza used an averaged potential approach to analyze the stability of the nonlinear system, also in the limiting case of high-frequency, small-amplitude base oscillation (Kapitza, 1951, 1965). For sinusoidal forcing, the linearized equations of motion of the vertically forced pendulum become Mathieu's equation (Jordan & Smith, 2007; Magnus & Winkler, 1979). Mathieu's equation contains two parameters, and the dependence of the stability of the system on the value of these parameters has been the subject of extensive study. Mathieu's equation is a special case of the 2nd-order linear periodic differential equation called Hill's equation. The *stability map* for Hill's equation, sometimes called an Ince–Strutt diagram, shows the regions of unstable and stable systems in the parameter space. These stability maps are often used to graphically illustrate the behavior of the Mathieu and Hill equations, but their application to control is less common. However the information available from the stability map is more comprehensive than the information provided by first-order averaging, making these diagrams potentially powerful tools for design. Subsequently we interpret the averaging approach in the context of the stability map, and show that direct use of the stability map can avoid drawbacks of first-order averaging methods.

Use of the stability map for control design can be extended to nonlinear and higher-order systems using Lyapunov's indirect method. This is illustrated below with two examples. The first highlights the ability of the proposed method to obtain a stabilizing input of arbitrary period on the classical vertically forced inverted pendulum. The second is control of a pitchfork bifurcation arising in a fourth-order system modeling an electrostatic MEMS actuator (Wickramasinghe & Berg, 2012a,b, 2013a). In this case averaging methods are inherently incapable of extending the region of stable operation of the actuator, whereas the approach based on the stability map succeeds.

A specific example of direct use of the stability map for control of higher-order systems can be found in Acheson (1993), where the approach is used to stabilize an inverted N-pendulum. Second-order, linear stability maps are the standard tool for design of quadrupole mass filters and quadrupole ion traps (Douglas, 2009; Hart-Smith & Blanksby, 2012; Lee et al., 2013; Titov, 1998). The successful practical application of second-order linear stability results to these complex system suggests a broader role for these results in control design. In other control-related work, Insperger has analyzed stability of PD control for a time-delayed inverted pendulum using stability maps (Insperger, 2011). Finally, the Mathieu and Hill equations arise in models of numerous other physics and engineering applications. These are too many to survey here, but a sampling may be found in McLachlan (1947).

The rest of this paper is organized as follows: Section 2 reviews the results for stability of the Hill and Mathieu equations, including generation of stability maps for the inputs considered subsequently. Section 3 applies averaging theory to Hill's equation, and examines the result using the stability map. This framework clearly shows the potential drawbacks of the averaging framework, and suggests that direct use of the stability map may avoid these problems. Section 4 shows how the stability map for Hill's equation may be used for design, for example, for stabilization by a control input of specified frequency. Section 5 discusses extension to nonlinear and higher-order systems, and presents two motivating examples. Finally, Section 6 presents conclusions and directions for future work.

Preliminary versions of some results from this paper appeared in Berg and Wickramasinghe (2013) and Wickramasinghe and Berg (2013b).

2. Stability of Hill's equation

Consider the second-order linear periodic system $\ddot{y} + 2\zeta\Omega\dot{y} + \left[-\Omega^2 + u(t)\right]y = 0$ with damping ratio $\zeta \geq 0$ and periodic, zero-mean, forcing function $u(t + T) = u(t)$. For use with averaging methods it is convenient to give the input the form $u(t) = \omega B_1\phi(\omega t)$, where $\phi(\xi)$ is a zero-mean function with period 2π, hence

$$\ddot{y} + 2\zeta\Omega\dot{y} + \left[-\Omega^2 + \omega B_1\phi(\omega t)\right]y = 0. \tag{1}$$

The stability of (1) may be studied by transformation to *Hill's equation*. First the damping term is eliminated by coordinate transformation (Magnus & Winkler, 1979)

$$y(t) = e^{-\zeta\Omega t}z(t) \tag{2}$$

to obtain

$$\ddot{z} + \left[-\Omega_\zeta^2 + \omega B_1\phi(\omega t)\right]z = 0 \tag{3}$$

where $\Omega_\zeta \triangleq \Omega\sqrt{1+\zeta^2}$. Applying time scaling $\tau = \omega t$ yields the standard form of Hill's equation (Magnus & Winkler, 1979),

$$z'' + [\alpha + \beta\phi(\tau)]z = 0, \tag{4}$$