Automatica 58 (2015) 51–59

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Shahshahani gradient-like extremum seeking[☆]

Jorge I. Poveda[a,1], Nicanor Quijano[b]

[a] Department of Electrical and Computer Engineering, University of California, Santa Barbara, USA

[b] Department of Electrical and Electronics Engineering, University of Los Andes, Bogotá, Colombia

ARTICLE INFO

Article history:
Received 2 May 2013
Received in revised form
9 October 2014
Accepted 24 April 2015
Available online 27 May 2015

Keywords:
Extremum seeking
Adaptive control
Optimization
Game theory
Nonlinear control systems

ABSTRACT

In this paper we present novel extremum seeking controllers designed to achieve online constrained optimization of uncertain input-to-output maps of a class of nonlinear dynamical systems. The algorithms, inspired by a class of evolutionary dynamics that emerge in the context of population games, generate on-line Shahshahani gradient-like systems able to achieve extremum seeking under simplex-like constraints on the positive orthant. The method attains semi-global practical convergence to the optimal point, even when the initial conditions are not in the feasible set, and it can be naturally adapted to address distributed extremum seeking problems in multi-agent systems where an all-to-all communication structure is not available. Potential applications include problems on distributed dynamic resource allocation, congestion and flow control in networked systems, as well as portfolio optimization in financial markets. Via simulations, we illustrate our results under different scenarios.

1. Introduction

Extremum seeking control (ESC) (Ariyur & Krstić, 2003) is a type of adaptive control designed to optimize the steady state input-to-output map of a nonlinear dynamical system in which, in principle, this map is not exactly known. Different types of ESCs have been proposed during the last years aiming to solve standard optimization problems (Nešić, Tan, Moase, & Manzie, 2010; Tan, Nešić, & Mareels, 2006), as well as to achieve online learning in non-cooperative and cooperative games (Frihauf, Krstić, & Başar, 2012; Kvaternik & Pavel, 2012; Stanković, Johansson, & Stipanović, 2000). In recent years, there has been interest to address the problem of extremum seeking with constraints. For instance, in Tan, Li, and Mareels (2013), the problem of extremum seeking with saturation constraints is addressed based on penalty functions and anti-windup techniques. Also, in Durr, Zeng, and Ebenbauer (2013), an extremum seeking algorithm based on a Lie-bracket approximation is shown to achieve convex optimization of static maps. Other works, such as DeHaan and Guay (2005), use penalty terms in the cost functions to integrate the constraints in the optimization problem and achieve extremum seeking. Nevertheless, most of the existing approaches are not suitable for the case when multivariable global constraints are included in the extremum seeking problem for dynamical systems, or when a lack of all-to-all communication structures preclude the implementation of centralized controllers. Since this type of optimization problems arises in several engineering, economics, and finance applications, there exists an increasing interest in designing adaptive and distributed algorithms for dynamical systems, that guarantee convergence to the optimal solution in a robust manner.

From this perspective, the main contribution of this paper is the introduction of novel extremum seeking controllers based on the framework of *population games* (Sandholm, 2010), which combine the main features of some evolutionary dynamics that emerge in the context of biological systems (Hofbauer & Sigmund, 1998), and the standard ESC. Since states with negative values are unrealistic in many practical applications we consider extremum seeking (ES) in the positive orthant, where the summation of the variables under control must be less or equal than a given positive value, allowing inequality constraints in the optimization problem. The ES dynamics proposed in this paper retain the optimality and adaptability properties of the classic evolutionary dynamics, as well as the robustness and non-model based characteristics of the standard ESC. Using the framework of population games we

☆ This work was supported in part by the Colombian Administrative Department of Science, Technology and Innovation (COLCIENCIAS), Programa Jóvenes Investigadores—2012. The material in this paper was partially presented at the 51st IEEE Conference on Decision and Control, December 10–13, 2012, Maui, Hawaii, USA and at the American Control Conference, June 17–19, 2013, Washington, USA. This paper was recommended for publication in revised form by Associate Editor Raul Ordóñez under the direction of Editor Miroslav Krstic. This work was performed while J.I. Poveda was with University of Los Andes, Bogotá, Colombia.

E-mail addresses: jipoveda@ece.ucsb.edu (J.I. Poveda), nquijano@uniandes.edu.co (N. Quijano).

1 Tel.: +1 805 8868162.

http://dx.doi.org/10.1016/j.automatica.2015.05.002

show that the ES dynamics guarantee convergence, in a semi-global practical sense, to the optimal solution of a constrained optimization problem with a global cost function, as well as in systems describing *strictly stable games* where a potential function does not exist. Additionally, we show that these dynamics can be naturally modified to achieve ES in multi-agent systems without an all-to-all communication structure.

The remainder of this paper is organized as follows. In Section 2 some preliminary concepts are presented. In Section 3 we introduce the ESC for constrained optimization of dynamical systems with a global potential function. Section 4 extends the results for systems where the potential function does not exist. Section 5 addresses the problem of distributed extremum seeking in multi-agent systems. Section 6 discusses some further remarks on the implementation of the algorithms, while Section 7 presents some numerical simulations. Finally, Section 8 ends with some conclusions.

2. Preliminaries

We denote by $\mathbb{N}_{>0}$ the set of positive integers, and by $\mathbb{R}^n$ the set of real numbers in the n-dimension, where $n \in \mathbb{N}_{>0}$. Also, we denote by $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ the set of non-negative real numbers and strictly positive real numbers, respectively, in the n-dimension. We use ∇f_i to represent the classical Euclidean gradient of a continuously differentiable function $f_i : \mathbb{R}^n \to \mathbb{R}^n$, and $\nabla_i f_i$ to represent the ith entry of ∇f_i, i.e., $\nabla_i f_i := \frac{\partial f_i(z)}{\partial z_i}$. Given a point $z \in \mathbb{R}^n$ the set $\mathcal{S}(z) := \{i \in \{1, \ldots, n\} : z_i > 0\}$ denotes the support of z. Given a set M we use $\overline{M}$ to denote its closure, and $\mathcal{S}_{z^*}(M) := \{z \in M : \mathcal{S}(z^*) \subset \mathcal{S}(z)\}$ to denote the set of points in M whose support contain the support of $z^* \in M$. The symbol $\mathbb{B}$ denotes a closed unit ball, $\rho\mathcal{B}$ a closed ball of radius $\rho > 0$, and $M + \rho\mathcal{B}$ the union of all sets obtained by taking a closed ball of radius ρ around each point in the set M. We use $\mathbf{1}$ to represent the column vector of appropriate dimensions with entries equal to 1, and $\Delta^n_r := \{z \in \mathbb{R}^n_{\geq 0} : \mathbf{1}^\top z = r\}$ to represent the simplex in the n-dimension, parameterized by the constant $r \in \mathbb{R}_{>0}$. We define $\text{int}(\Delta^n_r) := \{z \in \mathbb{R}_{>0} : \mathbf{1}^\top z = r\}$, $\bar{\Delta}^n_r := \{z \in \mathbb{R}^n_{\geq 0} : \mathbf{1}^\top z \leq r\}$, and the corresponding set $\text{int}(\bar{\Delta}^n_r)$. We use the acronyms AS and SPAS to refer to asymptotical stability (Khalil, 2002), and semi-global practical asymptotical stability (Tan et al., 2006; Teel, Moreau, & Nešić, 2003) respectively.

2.1. Population games and strictly stable games

Consider a population of entities playing a game, where each entity selects a pure strategy i from the set $\mathcal{H}^n = \{1, \ldots, n\}$, where $n \in \mathbb{N}_{>0}$. Let y_i be the normalized proportion of entities playing strategy i, such that the population state $y = [y_1, \ldots, y_n]^\top$ satisfies $y \in \Delta^n_r$, for some fixed $r > 0$. The payoff function associated to the ith strategy is defined as a real-valued continuously differentiable function $f_i : \Delta^n_r \mapsto \mathbb{R}$, and the vector of payoffs is defined as $f(y) := [f_1(y), \ldots, f_i, (y) \ldots, f_n(y)]^\top$. The interactions among the entities describe a population game, where two of the main equilibrium concepts are the Nash equilibrium (NE) and the globally evolutionarily stable state (GESS). The following definitions are adapted from Sandholm (2010).

Definition 1. For a population game defined in Δ^n_r, with vector of payoffs given by $f(y)$, the point y^* is a NE if $(y - y^*)^\top f(y^*) \leq 0$, for all $y \in \Delta^n_r$. If this inequality is satisfied uniquely by y^* we say that $\text{NE}(f) = \{y^*\}$.

Definition 2. For a population game defined in Δ^n_r, with vector of payoffs given by $f(y)$, the point y^* is a GESS if $(y - y^*)^\top f(y) < 0$, for all $y \in \Delta^n_r \backslash \{y^*\}$. If this strict inequality is satisfied we say that $\text{GESS}(f) = \{y^*\}$.

Note that according to Definitions 1 and 2, a GESS is a NE, but a NE is not necessarily a GESS. Also, it is a standard fact that every population game admits at least one Nash equilibrium. If this NE is also a GESS, then it is also unique (Sandholm, 2010).

In this paper we will focus on optimization and learning problems that can be analyzed using the framework of a special type of population games, named *strictly stable games*.

Definition 3. A population game in Δ^n_r, with vector of payoffs $f(y)$, is said to be strictly stable if it satisfies

$$(y - z)^\top (f(y) - f(z)) < 0, \quad \forall\, y, z \in \Delta^n_r. \tag{1}$$

Stable games have been studied in the context of transportation science (Dafermos, 1980; Smith, 1979), and more recently in the context of feedback control and passive systems (Fox & Shamma, 2013). Examples of strictly stable games include some types of symmetric normal form games, negative dominant diagonal games, and strictly concave potential games. For a detailed discussion about stable games the reader is referred to Sandholm and Hofbauer (2009).

Since in general we will be dealing with dynamical systems for which the function f is an unknown mapping defined in $\mathbb{R}^n$, and not only in Δ^n_r, we will say that a continuous vector function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the *strictly stable game condition in* Δ^n_r, if it satisfies Eq. (1). The following Lemma, adapted from Sandholm (2010), relates the concepts of NE, GESS, and strictly stable games.

Lemma 1. *Consider a strictly stable game defined in Δ^n_r, with vector of payoffs f. Then, its unique NE is also a GESS.*

2.2. Shahshahani gradients

The evolution in time of the population state $y(t)$ towards a NE or a GESS can be ruled by different dynamics known as evolutionary dynamics (Hofbauer & Sigmund, 1998). In the present work we consider the most recognized evolutionary dynamics, i.e., the replicator dynamics (RDs), which are given by

$$\dot{y}_i = \alpha\, y_i \left(f_i(y) - \bar{f}(y)\right), \quad \forall\, i \in \mathcal{H}^n, \tag{2}$$

where $\bar{f} := (1/Y) f^\top y$, and α and Y are positive constants. The replicator equation has been used in applications ranging from biology (Akin, 1990) and engineering (Ramirez-Llanos & Quijano, 2010), to portfolio optimization in financial markets (Bomze, 2005), for example. A particular feature of the replicator equation (2) is that, if the vector of payoffs f is selected as the classic Euclidean gradient vector of a smooth potential function J, it will describe a special type of gradient systems, named Shahshahani gradients.

Definition 4. Consider the manifold $M = \mathbb{R}^n_{>0}$ associated with the Riemannian metric $G(z) = g_{ij}(z)$, where $g_{ij} = \delta_{ij} \frac{\mathbf{1}^\top z}{z_i}$, $\delta_{ij} = 1$ only if $i = j$, and $\delta_{ij} = 0$, otherwise. Consider also a real-valued smooth function $J : M \mapsto \mathbb{R}$, whose derivative at the point z is given by the linear map $DJ : T_z M \mapsto \mathbb{R}$, where $T_z M$ is the tangent space of M at point z. Hence, there exists a unique gradient vector $\nabla_g J$ such that $\left\langle \eta, \nabla_g J \right\rangle_z^{G(z)} = DJ[\eta]$, for all $\eta \in T_z M$, where the operator $\langle \cdot, \cdot \rangle^G_z$, represents the inner product associated to the Riemannian metric $G(z)$ at point z, and $DJ[\eta] = \sum_{i=1}^N \frac{\partial J}{\partial z_i} \eta_i$. Then, $\nabla_g J$ is a Shahshahani gradient with potential function J, and $G(z)$ is the Shahshahani metric (Edalat, 2002).