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Revealing compactness of basins of attraction of multi-DoF dynamical systems

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Global properties of Multi-Degrees-of-Freedom (M-DoF) systems, in particular phase space organization, are largely unexplored due to the computational challenge requested to build basins of attraction. To overcome this problem, various techniques have been developed, some trying to improve algorithms and to exploit high speed computing, others giving up to possibility of having the exact phase space organization and trying to extract major information on a probability base. Following the last approach, this work exploits the method of "basin stability" (Menck et al., 2013) in order to drastically reduce the numerical effort. The probability of reaching the attractors is evaluated using a reasonable number of trials with random initial conditions. Then we investigate how this probability depends on particular generalized coordinate or a pair of coordinates. The method allows to obtain information about the basins compactness and reveals particular features of the phase space topology. We focus the study on a 2-DoF multistable paradigmatic system represented by a parametric pendulum on a moving support and model of a Church Bell. The trustworthiness of the proposed approach is enhanced through the comparison with the classical computation of basins of attraction performed in the full range of initial conditions. The proposed approach can be effectively utilized to investigate the phase space in multidimensional nonlinear dynamical systems by providing additional insights over traditional methods.

1. Introduction

The perception of how seemingly small causes are able to propagate towards grave effects has been noted by historians and others over several centuries, "for want of a nail ... a kingdom was lost" [1]. Fairly well understood the possibility of unforeseen consequences, scientists' curiosity turned on investigation of triggers acts and on their abatement. Back to the 19th century, the study around the sensitive dependence on initial conditions is anticipated finding its influence in the complexity of trajectories in the three-body problem [2], but only in mid 1900s, flourishing studies of differential equations were able to make rigorous order in the apparent unpredictability of dynamical systems [3–5]. While for a linear finite-dimensional system a uniform exponential decay to the sole equilibrium is observed, nonlinear systems may be character-

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ized by a multistable behaviour [6]. As consequence of possessing a variety of attractors, a nonlinear dynamic system exhibits complex and interesting scenarios of response [7].

With the aim to properly predict the response behaviour, the identification of both existence and stability of attractors does not suffice. This represents an unfortunate condition since numerical tools have turned their determination into a straightforward operation [8,9]. Conversely, what is required to identify is the structure of the phase space. This implies the analysis of the shape, size and boundaries of the basin (or domain) of attraction related to each attractor. Basins of attraction are, by definition, subsets of the phase space, i.e. sets of initial conditions, bringing the system to a specific attractor. Labelling the phase space is the key to map all the possible outcomes of a dynamical system. As a matter of fact, the fractality of basins of attraction is one of the motivations behind the sensitivity to initial conditions, i.e. one of the roots of unpredictability or catastrophic consequences. The mere estimation of the stability in the classical, or Lyapunov [10], meaning is not sufficient for the practical stability in the engineering sense for mulistable systems for mulistable systems. Indeed, from a local point of view, a stable solution may be not visible in practice. If the basins associated with the attractor is small and compact enough, the system could not be able to accommodate noise and perturbations becoming practically unsafe [11].

Whereas in some cases approximating analytical techniques are available to study the local dynamics [12], the determination of domains of attraction requires significant computational resources [13,14]. Thus, beside the recognized usefulness and potentiality, examples of global dynamics approach in high-dimensional systems are still scarce [15]. The analysis essentially relies on 2-dimensional (2D) sections of the n-dimensional basins of attraction. Few examples are: (i) the projections of a 4D phase space to estimate the dynamic integrity of a parametrically excited cylindrical shell [16]; (ii) instability phenomena under parametric excitation of flexural modes described by use of 2D section in [17], (iii) a proper indication of what occurs in the large dimensional phase space of the three-dimensional model describing the throw of a die obtained by means of 2D basins of attraction in [18].

Although new numerical techniques has been recently introduced [13,19], the computation of basins of attraction in large-dimensional systems remain computationally expensive. An alternative (approximated) approach that avoids to scan the complete set of all the initail conditions is represented by the "basin stability" method [20]. The equations of motion are repeatedly integrated with random initial conditions and for each trial the final attractor is identified. In a multistable scenario, a chosen set of initial conditions generates the probability of occurrence of different attractors [21,22]. It has to be stressed that the aforementioned analysis, when the phase space topology is highly riddled or fractal, i.e. it presents a low compactness, has to be extended to the full ensemble of initial conditions. This is done by making use of the classical computation algorithms for basins of attraction, e.g. the integration of a grid of points [23] or cell-mapping methods [24] and their related evolutions [13,25]. The accuracy of solutions detection depend on the number of grid points. If the solution is hidden or rare attractor [26] with really narrow range of stability or it is Milnor attractor [27] we can miss it in calculation. However, if the grid is dense and we cannot detect the solution its importance, form practical point of view, is marginal. In [22] double pendulum system has been analysed. Overall, 172,500 trials (integration of system equations with different, random initial conditions) were performed. Some solutions were reached only once per 172,500 trials, thus we can classify them as hidden attractors, but they can be neglect in practical analysis.

This work presents the analysis of a parametric pendulum based on the basin stability method along with a classical analysis based on basins of attraction. In particular, for an improved reliability, the domains are not collected in bidimensional sections, but the full 4D phase space is investigated. This is doable only thanks to the use of parallel computation that permits a reasonable elaboration time and to avoid memory overflows [28]. The parametric pendulum results a perfect candidate as a paradigm model to demonstrate the usefulness of the proposed approach. It shows a rich dynamical behaviour (from simple periodic oscillation to complex chaos [29,30]) Remarkable achievements are the recent deploy of parametric pendulum-based devices toward commercializable wave energy converters [31–34].

The paper considers the forced oscillations of a damped pendulum coupled with Duffing oscillator. The mechanical 2-DoF of the system are the vertical translation of the harmonic oscillator and the rotation of the suspended pendulum; they correspond to a system with four dimensional phase space. The governing equations of motion are obtained in Section 2 by using a Lagrangian approach. Here we introduce the model which we use to demonstrate the idea of the work. Section 3 reports numerical results in the case of two and six coexisting stable solutions, respectively. The possible application of proposed idea is presented in Section 4 based on model of Church Bell. Finally, Section 5 rounds up the paper with our conclusions.

2. Model of a Duffing oscillator with a suspended pendulum

The analyzed system is shown in Fig. 1. It consists of a Duffing oscillator with a suspended pendulum. The Duffing system is excited with harmonic force and moves in vertical direction. The position of mass M is given by coordinate x and the angular displacement of pendulum (position of the mass m) is given by angle φ. The equations of motion can be derived using Lagrange equations of the second type. The kinetic energy T, potential energy V and Rayleigh dissipation D are given respectively by the following equations:

$$T = \frac{1}{2}(M+m)\dot{x}^2 - ml\dot{x}\dot{\varphi}\sin\varphi + \frac{1}{2}ml^2\dot{\varphi}^2, \tag{1}$$