Automatica 58 (2015) 160–166

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Linear analysis of the vectorial network model in the presence of leaders ☆

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ARTICLE INFO

Article history:
Received 6 March 2014
Received in revised form
16 January 2015
Accepted 13 May 2015
Available online 6 June 2015

Keywords:
Consensus
Leader–follower
Networks
Polarization
Stochastic stability
Synchronization

ABSTRACT

In this work, we study the collective behavior of a multi-agent system of stochastically interacting leaders and followers. In this time-varying network, the nodes update their states through a noisy interaction with a randomly selected subset of neighbors, including leaders whose average orientation is not updated in time. By linearizing the system dynamics in the vicinity of the leaders' common trajectory, we establish a toolbox of closed-form expressions that aid in the understanding of the influence of noise, number of connected neighbors, network size, and proportion of leaders on the group coordination.

1. Introduction

Collective behavior is a common phenomenon in several biological groups, such as bird flocks, fish shoals, and insect swarms (Krause & Ruxton, 2002; Smelser, 2011; Sumpter, 2006). This phenomenon is often the result of local interactions among the group's individuals (Cavagna et al., 2010; Couzin, Franks, & Levin, 2005) and is manifested in the form of highly coordinated actions that have inspired the solution of several engineering problems in the areas of synchronization, including neural networks (Ijspeert, 2008; Kennedy, Eberhart et al., 1995) and communications (Dorigo, Di Caro, & Gambardella, 1999; Wang & Slotine, 2006).

Biological groups can be modeled as a network of coupled dynamical systems, in which the nodes correspond to the individuals and the edges represent the interactions among them (Abaid & Porfiri, 2010). Studies on biological groups indicate that knowledge about the environment may significantly vary among individuals, whereby some may have access to information that is not available to others (Couzin et al., 2005). Often, a few informed individuals may act as group leaders by influencing the uninformed conspecifics, and thus modulating the overall collective response (Dyer, Johansson, Helbing, Couzin, & Krause, 2009). Upon modeling the group as a network of dynamical systems, the emergence of leader–follower relationships is analogous to pinning control (Grigoriev, Cross, & Schuste, 1997; Liu, Slotine, & Barabasi, 2011).

An effective network model used to describe collective behavior is the vectorial network model (VNM), introduced in Ref. Aldana, Dossetti, Huepe, Kenkre, and Larralde (2007) as a simplification of the classical Vicsek model (Czirók & Vicsek, 2000; Vicsek, Czirók, Ben-Jacob, Cohen, & Shochet, 1995), which has found extensive applications in engineering and biology to describe the emergent behavior of social animals and multi-vehicle teams (Vicsek & Zafeiris, 2012). The VNM consists of N interacting agents represented by two-dimensional vectors expressed as complex numbers $v_1 = \mathrm{e}^{\mathrm{I}\theta_1}, \ldots, v_N = \mathrm{e}^{\mathrm{I}\theta_N}$, where $\theta_i, i = 1, \ldots, N$ are the angles defining the orientation of each agent, and I is the imaginary unit. In a discrete-time setting, the agents interact with a subset of K randomly selected agents, from which they update their orientation. Thus, each agent determines its orientation at the next time step from the average of the two-dimensional vectors corresponding to its neighbors, subject to noise. The random selection of neighbors in this model is executed at each time step, and prevents the coupling between the orientation of the agents and their local density as in the Vicsek model (Chaté, Ginelli, Grégoire, & Raynaud, 2008; Czirók & Vicsek, 2000; Pimentel, Aldana, Huepe, & Larralde, 2008; Vicsek et al., 1995). Similar

☆ This work was supported by the National Science Foundation under Grant # CMMI-0745753 and GK-12 Fellows Grant # DGE-0741714. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Wei Ren under the direction of Editor Christos G. Cassandras.

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http://dx.doi.org/10.1016/j.automatica.2015.05.018

metric-free models have been recently presented in Refs. Ballerini et al. (2008); Chou, Wolfe, and Ihle (2012).

To model, for the first time, the presence of informed individuals in the VNM, we assume that a proportion ϕ of the agents are leaders. Without loss of generality, we hypothesize that the nodes $i = 1, \ldots, N_f = (1-\phi)N$ are followers that update their orientation based on the state of K neighbors, and the $N_l = \phi N$ remaining nodes, $i = N_f + 1, \ldots, N$, are leaders that share a common reference state θ_0. With respect to biological groups, such common orientation may be associated with a migration route or a food source (Couzin et al., 2005). At each time $k \in \mathbb{Z}^+$, the node i, $i = 1, \ldots, N$, computes the following average vector:

$$U_i(k) = \begin{cases} \dfrac{1}{K}\displaystyle\sum_{j=1}^{K} v_{i_j}(k), & \text{if } i \text{ is a follower,} \\ v_i, & \text{if } i \text{ is a leader,} \end{cases} \tag{1}$$

where $i_1, \ldots, i_K$ are the agents connected to node i; this set can also include node i.

The orientations of the agents at any time $k \in \mathbb{Z}^+$ can be written in a compact form as $\theta(k) = \left[\theta^f(k)^{\mathrm{T}}, \theta^l(k)^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\theta^f(k) = \left[\theta_1(k), \ldots, \theta_{N_f}(k)\right]^{\mathrm{T}}$ is the state of the followers and $\theta^l(k) = \left[\theta_{N_f+1}(k), \ldots, \theta_N(k)\right]^{\mathrm{T}}$ is the state of the leaders. Agents are also subject to intrinsic noise, and the dynamics of the system is given by

$$\theta_i(k+1) = \begin{cases} \arg\{U_i(k)\} + \eta\zeta_i(k), & \text{if } i \text{ is a follower,} \\ \theta_0 + \eta\zeta_i(k), & \text{if } i \text{ is a leader,} \end{cases} \tag{2}$$

where $0 \le \eta \le 1$ quantifies the noise intensity, $\arg(\cdot)$ is the argument of a complex number, $\zeta_1(k), \ldots, \zeta_N(k)$ are independent and identically distributed (i.i.d.) random variables with common uniform random variable taking values in $[-\pi, \pi]$.

In the absence of leaders, studies on the VNM have investigated the effects of group size N, number of connected neighbors K, and noise intensity η on the system coordination (Aldana et al., 2007; Pimentel et al., 2008; Porfiri, 2014). Specifically, in the thermodynamic limit $N \to \infty$, it has been shown that the system exhibits a continuous order–disorder phase transition (Aldana et al., 2007; Pimentel et al., 2008). In other words, the vectors synchronize toward a common orientation for $\eta = 0$, and the degree of synchronization smoothly decreases as noise increases, until a phase transition to a completely disordered state is observed. The critical noise at which such phase transition occurs is controlled by K, and a closed-form solution for $K \to \infty$ has been presented in Ref. Pimentel et al. (2008).

To investigate the VNM beyond the thermodynamic limit and thus offer valuable insight into the coordination of biological groups, an alternative methodology has been proposed in Ref. Porfiri (2014). Therein, the system dynamics is linearized for $\eta \ll 1$, resulting in a linear stochastic consensus problem (Cao, Yu, Ren, & Chen, 2013; Hatano & Mesbahi, 2005; Huang & Manton, 2010; Kar & Moura, 2008; Patterson, Bamieh, & El Abbadi, 2010; Pereira & Pages-Zamora, 2010; Porfiri & Stilwell, 2007; Tahbaz-Salehi & Jadbabaie, 2008; Wu, 2006; Zhou & Wang, 2009). By adapting the algebraic analysis proposed for mean square consensus in Refs. Abaid, Igel, and Porfiri (2012); Abaid and Porfiri (2011, 2012), a toolbox of closed-form results for the system dynamics are established. Numerical findings from independent simulations suggest that such analytical expressions accurately reproduce the system response for noise levels in the range $\eta \lesssim 0.5$.

In this work, we extend the methodology presented in Ref. Porfiri (2014) to investigate the effect of group leaders on the coordination of the VNM. We specifically focus on the linear stochastic dynamics stemming from (1) and (2) for $\eta \ll 1$ and on small variations of the orientation of the followers with respect to the leaders. Hence, we define the disagreement between the angle of an agent i and the reference angle θ_0 by $\varepsilon_i(k) = \theta_i(k) - \theta_0$. For small variations $\varepsilon_1(k), \ldots, \varepsilon_N(k)$, we linearize the model in Eq. (2) to obtain the linear stochastic dynamics of the disagreement, that is,

$$\varepsilon_i(k+1) = \begin{cases} \dfrac{1}{K}\displaystyle\sum_{j=1}^{K} \varepsilon_{i_j}(k) + \eta\zeta_i(k), & \text{if } i \text{ is a follower,} \\ \eta\zeta_i(k), & \text{if } i \text{ is a leader.} \end{cases} \tag{3}$$

For convenience, we write the disagreement vector as $\varepsilon(k) = \left[\varepsilon^f(k)^{\mathrm{T}}, \varepsilon^l(k)^{\mathrm{T}}\right]^{\mathrm{T}}$ with $\varepsilon^f(k) = \left[\varepsilon_1(k), \ldots, \varepsilon_{N_f}(k)\right]^{\mathrm{T}} \in \mathbb{R}^{N_f}$ and $\varepsilon^l(k) = \left[\varepsilon_{N_f+1}(k), \ldots, \varepsilon_N(k)\right]^{\mathrm{T}} \in \mathbb{R}^{N_l}$; and, similarly, we write the intrinsic noise vector as $\zeta(k) = \left[\zeta^f(k)^{\mathrm{T}}, \zeta^l(k)^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\zeta^f(k) = \left[\zeta_1(k), \ldots, \zeta_{N_f}(k)\right]^{\mathrm{T}} \in \mathbb{R}^{N_f}$ and $\zeta^l(k) = \left[\zeta_{N_f+1}(k), \ldots, \zeta_N(k)\right]^{\mathrm{T}} \in \mathbb{R}^{N_l}$, respectively. Using this compact notation, the linear stochastic system (3) can be rewritten as

$$\varepsilon(k+1) = \mathcal{C}(k)\varepsilon(k) + \eta\zeta(k), \tag{4}$$

where $\mathcal{C}(k)$ are the state matrices. Similar to Ref. Porfiri (2014), $\mathcal{C}(k)$ are i.i.d. matrices with common random variable $\mathcal{C}$, which is defined so that its first N_f rows are i.i.d. vectors with K randomly selected entries taking value $\frac{1}{K}$ while all other entries equal 0. Similar to most of the classical studies on stochastic consensus protocols (Hatano & Mesbahi, 2005; Porfiri & Stilwell, 2007; Tahbaz-Salehi & Jadbabaie, 2008), this matrix is non-negative; however it is generally non-symmetric and, most importantly, its entries are not independent random variables. We further note that the problem shares similarities with Ref. Abaid and Porfiri (2012), which addresses leader–follower consensus in numerosity-constrained networks. However, differently than Ref. Abaid and Porfiri (2012), we consider a noisy consensus protocol which stems from the linearization of a nonlinear model. The resulting state matrix differs from the numerosity-constrained network model in Ref. Abaid and Porfiri (2012), whereby each agent is not required to utilize its past orientation during the updating process. As a result, the diagonal entries of the state matrix may vary in time, and the agents interact with K or $K-1$ others.

In what follows, we refer by I_N the $N \times N$ identity matrix; the $N \times 1$ vector of all ones is referred by 1_N; the $N \times 1$ vector of all zeros is 0_N and $0_{N\times M}$ is the $N \times M$ matrix of all zeros; the Kronecker product of two matrices A and B is $A \otimes B$; the operation $\mathrm{vec}(\cdot)$ is the vectorization of a matrix by stacking its columns; the matrix transposition is referred by a superscript T; and $\mathbf{E}[\cdot]$ refers to the expected value of a random variable.

2. Asymptotic behavior

2.1. Mean square analysis

Following Abaid et al. (2012); Abaid and Porfiri (2011, 2012), we consider the second moment matrix, sometimes referred to as autocorrelation, $\varXi(k) = \mathbf{E}\left[\varepsilon(k)\varepsilon(k)^{\mathrm{T}}\right]$ to investigate the mean square behavior of the stochastic system (3). Notably, the trace of such a matrix measures the mean square deviation, that is, $\delta(k) = \mathbf{E}\left[\|\varepsilon(k)\|^2\right] = \mathrm{vec}(I_N)^{\mathrm{T}}\mathrm{vec}\left(\varXi(k)\right)$.

Following the decomposition in Ref. Abaid and Porfiri (2012), the leaders do not update their state on the basis of their neighbors, and the linearized model is reduced to contain only the followers' states. Hence, we write the first N_f rows of $\mathcal{C}(k)$ as the augmented matrix $\left[\mathcal{C}^{ff}(k)|\mathcal{C}^{fl}(k)\right]$, where $\mathcal{C}^{ff}(k) \in \mathbb{R}^{N_f\times N_f}$ and $\mathcal{C}^{fl}(k) \in \mathbb{R}^{N_f\times N_l}$ are i.i.d. matrices with common random variables $\mathcal{C}^{ff}$ and $\mathcal{C}^{fl}$, respectively. Thus, we have

$$\mathcal{C} = \left[\begin{array}{c|c} \mathcal{C}^{ff} & \mathcal{C}^{fl} \\ \hline 0_{N_l\times N_f} & 0_{N_l\times N_l} \end{array}\right]. \tag{5}$$