Automatica 58 (2015) 39–42

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

On constructing Lyapunov functions for multi-agent systems[☆]

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ARTICLE INFO

Article history:
Received 6 November 2014
Received in revised form
4 February 2015
Accepted 20 April 2015
Available online 27 May 2015

Keywords:
Graph dependent Lyapunov equation
Lyapunov function
Multi-agent system
Nonnegative graph

ABSTRACT

Lyapunov equations that depend on communication graph topologies provide building blocks of Lyapunov functions, which play an important role in controller design and stability analysis of multi-agent systems. However, construction of such a Lyapunov equation in some published works has a flaw or requires unnecessarily strong graph conditions. This paper presents several choices of Lyapunov equations over various graph topologies.

1. Introduction

Cooperative control of multi-agent systems has attracted a lot of attention in the control community for the past few years, with some representative works being Jadbabaie, Lin, and Morse (2003); Olfati-Saber and Murray (2004); Ren and Beard (2005), etc. Stability analysis of multi-agent systems must take into account the connectivity property of the graph, i.e., the way in which the individual systems communicate. The importance in cooperative control stability analysis of constructing a Lyapunov function that depends on communication graph properties is emphasized in Das and Lewis (2010); Lewis, Zhang, Hengster-Movric, and Das (2014); Meng, Zhao, and Lin (2013); Qu (2009); Zhang and Lewis (2012); Zhang, Lewis, and Qu (2012), and elsewhere in the literature.

For cooperative tracking problem, a graph dependent Lyapunov equation was proposed in Das and Lewis (2010, Lemma 2) when the graph containing all follower nodes is strongly connected. The condition was relaxed in Zhang et al. (2012) to allow the graph to have a spanning tree, and further to allow the augmented graph to have a spanning tree (Zhang & Lewis, 2012). It was recently noticed that in the Lyapunov equation proposed in Das and Lewis (2010, Lemma 2), the construction of the diagonal matrix P cannot guarantee a positive definite matrix Q as claimed. A counterexample was given in Su, Lin, and Garcia (2014), which proposed a correct choice of positive definite diagonal matrix P for strongly connected digraphs. However, the graph condition in Su et al. (2014) is unnecessarily strong, and can be relaxed. Also, a graph dependent Lyapunov equation proposed in Meng et al. (2013) requires the graph to be strongly connected and detailed balanced. This is rather restricted. Considering the importance of graph dependent Lyapunov equations in stability analysis of multi-agent systems, this paper aims to provide a thorough treatment of this issue and lists several Lyapunov equations over different graph topologies, which may be used as building blocks of Lyapunov functions for multi-agent systems.

Notations. For notational convenience, 0 denotes either zero scalar, zero vector or zero matrix, according to the context. A vector $x = [x_1, \ldots, x_n]^T$ is positive, written as $x > 0$, if $x_i > 0$ for all i; and nonnegative, written as $x \geq 0$, when all its entries are nonnegative.

☆ This work was supported in part by the National Natural Science Foundation of China under grants 61304166, 61433011 and 61473005, the Research Fund for the Doctoral Program of Higher Education under Grant 20130184120013, and by US National Science Foundation under grants ECCS-1308928 and ECCS-1405173. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Carlo Fischione, under the direction of Editor André L. Tits.

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http://dx.doi.org/10.1016/j.automatica.2015.05.006
0005-1098/

Matrix $A > 0$ ($A \geq 0$) means that A is positive definite (positive semi-definite); while a positive (nonnegative) matrix is denoted as $A \succ 0$ ($A \succeq 0$), where all its entries are positive (nonnegative). The spectral radius of matrix A is $\rho(A)$. The empty set is denoted as $\varnothing$. The vector with all entries being ones is $\underline{1}$, and the identity matrix with appropriate dimensions is denoted as I.

2. Preliminaries

Consider a multi-agent system consisting of N agents (nodes). The communication network among these agents is modeled by a directed graph $\mathcal{G} = \{\mathcal{V}, A\}$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is the node set and $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix of the graph, with a_{ij} being the component in the ith row and jth column of A. We consider nonnegative graphs, i.e., if there is an edge from node j to node i, denoted as (v_j, v_i), then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. Node j is called a neighbor of node i, if $a_{ij} > 0$. Denote the neighbor set of node i as $\mathcal{N}_i = \{j \mid a_{ij} > 0, j = 1, 2, \ldots, N\}$. We assume the graph is simple, i.e., $a_{ii} = 0$. A graph is undirected if $a_{ij} = a_{ji}$ for all i and j, and directed otherwise. The graph Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i \\ \sum_{j=1}^{N} a_{ij}, & j = i. \end{cases}$$

A directed path from node i to node j is a sequence of adjacent edges $(v_i, v_l), (v_l, v_p), \ldots, (v_q, v_j)$. A graph *has a spanning tree* if there is a root node that has a path to every other node. A graph is *strongly connected* if there is a directed path between every ordered pair of nodes. A graph $\mathcal{G}(A)$ is strongly connected if and only if its adjacency matrix A is irreducible (Berman & Plemmons, 1994).

Definition 1 (*Reducibility/Irreducibility Qu, 2009*)**.** A nonnegative matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ with $N \geq 2$ is said to be reducible if the set of its indices $\Omega = \{1, \ldots, N\}$ can be divided into two disjoint nonempty sets $\Omega_p = \{i_1, \ldots, i_p\}$ and $\Omega_q = \{j_1, \ldots, j_q\}$, satisfying $\Omega_p \cup \Omega_q = \Omega$ and $\Omega_p \cap \Omega_q = \varnothing$, such that $a_{i_\alpha j_\beta} = 0$ for all $\alpha = 1, \ldots, p$ and $\beta = 1, \ldots, q$. Matrix A is irreducible if it is not reducible.

It is widely recognized that Laplacian matrix L plays a fundamental role in stability analysis of multi-agent system. The Laplacian matrix for a nonnegative graph is a singular M-matrix (Qu, 2009). For easy reference, some background of M-matrix and nonnegative matrix are presented here, which will be used in the main results of this paper.

Let $\mathcal{Z}$ be a set of square matrices with nonpositive off-diagonal entries, i.e.,

$$\mathcal{Z} \triangleq \{A = [a_{ij}] \in \mathbb{R}^{n\times n} \mid a_{ij} \leq 0, \forall i \neq j\}.$$

M-matrix is a subclass of $\mathcal{Z}$, and is defined as below.

Definition 2 (*M-matrix Berman & Plemmons, 1994*)**.** A square matrix A is an M-matrix, if it can be expressed in the form

$$A = sI - C$$

for some nonnegative matrix $C \succeq 0$ and $s \geq \rho(C)$.

An M-matrix $A = sI - C$ is irreducible (reducible), if the nonnegative matrix C is irreducible (reducible).

Definition 3 (*Singular/Nonsingular M-matrix Berman & Plemmons, 1994*)**.** A square matrix A is a singular M-matrix, if it is an M-matrix with $s \geq \rho(C)$; it is a nonsingular M-matrix, if $s > \rho(C)$.

The next result is the well-known Perron–Frobenius theorem, which is a fundamental result on nonnegative matrices.

Lemma 1 (Qu, 2009)**.** *Let A be a nonnegative square matrix.*

(a) *Then $\rho(A) \geq 0$ is an eigenvalue and there exists a nonnegative vectors x, such that $Ax = \rho(A)x$.*
(b) *If $A \succ 0$, then $\rho(A) > 0$ is a simple eigenvalue, and its eigenvector is positive.*
(c) *If A is irreducible, then there exists a positive vector x such that $Ax = \rho(A)x$, and $\rho(A) > 0$ is a simple eigenvalue of A.*

Lemma 2 (Berman & Plemmons, 1994)**.** *A square matrix A ($A \in \mathcal{Z}$) is a nonsingular M-matrix, if and only if one of the following equivalent conditions holds:*

(a) *There is a positive vector $x > 0$ such that $Ax > 0$.*
(b) *There is a positive vector $y > 0$ such that $A^T y > 0$.*
(c) *All the eigenvalues of A have positive real parts.*
(d) *A is nonsingular and A^{-1} is nonnegative.*

3. Constructing Lyapunov functions on graphs

For cooperative control of multi-agent systems, two extensively studied problems are the leaderless consensus problem (also known as consensus problem, or cooperative regulation problem) and the leader-following consensus problem (or cooperative tracking problem). For the former problem, all agents play an equal role and tend to reach a consensus. For the latter one, there is a leader node, labeled v_0, whose behavior is not affected by the follower nodes $v_1, \ldots, v_N$, and all followers are controlled to track the leader node (Lewis et al., 2014). In this section, we present a variety of Lyapunov equations on graphs for these two fundamental problems and show briefly how they relate to Lyapunov functions.

3.1. Lyapunov equation for leaderless consensus problems

Proposition 1 (Qu, 2009; Zhang et al., 2012)**.** *Suppose that the graph $\mathcal{G}$ is strongly connected. Let $p = [p_1, \ldots, p_N]^T > 0$ be a left eigenvector of the Laplacian matrix L associated with the eigenvalue 0, i.e., $L^T p = 0$. Define*

$$P = \operatorname{diag}(p_i),$$
$$Q = PL + L^T P. \tag{1}$$

Then $P > 0$ and $Q \geq 0$.

The existence of such a positive left eigenvector p is guaranteed by Lemma 1. A normalized p is usually chosen such that $L^T p = 0$ and $p^T \underline{1} = 1$. Proposition 1 is proved in Zhang et al. (2012) by introducing the concept of generalized Laplacian potential.

The Lyapunov equation (1) can be used to build a Lyapunov function. Take for example the consensus problem of multi-agent system with single-integrator dynamics

$$\dot{x}_i = u_i, \quad i = 1, 2, \ldots, N.$$

Adopting the well known consensus protocol (Olfati-Saber & Murray, 2004)

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)$$

leads to the closed-loop system

$$\dot{x} = -Lx, \tag{2}$$

where $x = [x_1, \ldots, x_N]^T$. It is clear that $V = x^T Px$ can serve as a Lyapunov function for system (2), where P solves the Lyapunov equation (1).

Remark 1. When L is an irreducible singular M-matrix in the general sense, not necessarily the Laplacian matrix of a graph, a general Lyapunov equation is provided in Qu (2009, Theorem 4.31) as

$$P = \operatorname{diag}(y_i/x_i),$$
$$Q = PL + L^T P,$$

where $x = [x_1, \ldots, x_N]^T$ and $y = [y_1, \ldots, y_N]^T$ are the positive right and left eigenvectors of L associated with the eigenvalue 0.