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Brief paper

Online constraint removal: Accelerating MPC with a Lyapunov function☆

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ABSTRACT

We show how to use a Lyapunov function to accelerate MPC for linear discrete-time systems with linear constraints and quadratic cost. Our method predicts, in the current time step, which constraints will be inactive in the next time step. These constraints can be removed from the online optimization problem of the next time step. The criterion for the detection of inactive constraints is based on the decrease of the Lyapunov function along the trajectory of the controlled system. The criterion is simple, easy to implement in existing MPC algorithms, and its computational cost is small.

1. Introduction

Model predictive control (MPC) is an established, practically relevant method for the control of constrained multivariable systems. MPC is computationally expensive, because an optimal control problem must be solved in each time step. For a discrete-time linear system with linear constraints and quadratic cost the optimal control problem is a quadratic program (QP) that is parametrized by the current state of the system. It is known that the solution to this parametric QP is a continuous piecewise affine control law $u(x)$ on a polytopic partition of the state space (Bemporad, Morari, Dua, & Pistikopoulos, 2002; Seron, Goodwin, & DeDona, 2003). Despite ongoing efforts to improve the algorithms for the calculation (Columbano, Fukuda, & Jones, 2009; Gupta, Bhartiya, & Nataraj, 2011; Mönnigmann & Jost, 2012; Patrinos & Sarimveis, 2010) and fast evaluation (Bayat, Johansen, & Jalali, 2011; Mönnigmann & Kastsian, 2011; Tøndel, Johansen, & Bemporad, 2003) of these piecewise affine control laws, they can only be calculated and used for small systems with short horizons. Thus, for medium or large systems online methods remain the only viable choice.

The discovery of the structure of the solution $u(x)$ prompted research on how to use this structure to accelerate online MPC algorithms. Ferreau, Bock, and Diehl (2008) predict the active set occurring in the next step. Pannocchia, Rawlings, and Wright (2007) and Pannocchia, Wright, and Rawlings (2011) enumerate the active sets which occurred most frequently during previous operation and store the optimal solution parameters *only* for those active sets. Jost and Mönnigmann (2013a,b) calculate state space *regions of activity* for each constraint offline and use this information online to remove inactive constraints from the QP. In this paper, we also accelerate the online MPC computation by removing constraints from the QP that can be inferred to be inactive before actually solving the QP. In contrast to earlier approaches (Jost & Mönnigmann, 2013a,b) our approach is neither based on the explicit solution nor an approximation thereof. We show that the cost function of the MPC problem can be used to bound the optimal solution for the next time step, if the cost function is a Lyapunov function for the controlled system. The bound on the optimal solution for the next time step only depends on information available at the current time step. While somewhat conservative, this bound can be used to remove some inactive constraints in the next time step, thus simplifying the QP. We stress the proposed method does not just remove constraints from the QP that can never become active, but the set of removed constraints is a function of the current state and thus a function of time.

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We introduce the problem class in Section 2. Section 3 establishes the main results described above. An example is discussed in Section 4, and an outlook is given in Section 5.

2. Problem statement and assumptions

Consider a discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, where the pair (A, B) is stabilizable. Assume the system (1) is subject to input and state constraints

$$u(t) \in \mathbb{U} \subset \mathbb{R}^m, \qquad x(t) \in \mathbb{X} \subset \mathbb{R}^n \tag{2}$$

for all times $t \in \mathbb{N}$, where $\mathbb{X}$ and $\mathbb{U}$ are compact full-dimensional polytopes that contain the origin in their interiors.

For any initial state $x \in \mathbb{X}$, consider the finite horizon optimal control problem

$$\begin{aligned} \mathbb{P}(x): \quad & \min_{U,X} V_f(x(N)) + \sum_{k=0}^{N-1} \ell(x(k), u(k)) \quad \text{s.t.} \\ & x(k+1) = Ax(k) + Bu(k), \quad k = 0, \ldots, N-1 \\ & x(0) = x, \\ & x(k) \in \mathbb{X}, \quad k = 1, \ldots, N-1, \\ & x(N) \in \mathbb{X}_f, \\ & u(k) \in \mathbb{U}, \quad k = 0, \ldots, N-1, \end{aligned} \tag{3}$$

where $U = \left(u'(0), \ldots, u'(N-1)\right)'$, $X = \left(x'(1), \ldots, x'(N)\right)'$, $\mathbb{X}_f \subseteq \mathbb{X}$ is a polyhedral terminal set that contains the origin in its interior, and $V_f(x) = \frac{1}{2}x'Px$, $\ell(x,u) = \frac{1}{2}(x'Qx + u'Ru)$, where $P \in \mathbb{R}^{n\times n}$, $P \succ 0$, $Q \in \mathbb{R}^{n\times n}$, $Q \succeq 0$ and $R \in \mathbb{R}^{m\times m}$, $R \succ 0$ are the weighting matrices on the terminal state, the states and the inputs, respectively. By eliminating the state variables with (1), the quadratic program (3) can equivalently be written in the form

$$\min_U V(x, U) \quad \text{s. t. } GU - w - Ex \le 0, \tag{4}$$

where

$$V(x, U) = \frac{1}{2}\begin{pmatrix} x' & U' \end{pmatrix}\begin{pmatrix} Y & F \\ F' & H \end{pmatrix}\begin{pmatrix} x \\ U \end{pmatrix} \tag{5}$$

$$= \frac{1}{2}x'Yx + U'F'x + \frac{1}{2}U'HU, \tag{6}$$

$H \in \mathbb{R}^{mN\times mN}$, $Y \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times mN}$, $G \in \mathbb{R}^{q\times mN}$, $E \in \mathbb{R}^{q\times n}$, $w \in \mathbb{R}^q$, and where q denotes the number of inequality constraints in (3) and (4). It can be shown that $Y' = Y$, $H' = H$ and $H \succ 0$, if $R \succ 0$, $P \succ 0$ and $Q \succeq 0$. Consequently, (4) is a strictly convex quadratic program. This implies the solution to (4), and equivalently to (3), is unique if it exists. We note for later use that (6) can be rewritten as $V(x,U) = \frac{1}{2}(U + H^{-1}F'x)'H(U + H^{-1}F'x) + \frac{1}{2}x'Yx - \frac{1}{2}x'FH^{-1}F'x$ by completing the squares. For brevity we write this expression for $V(x, U)$ as

$$V(x, U) = \frac{1}{2}\|U + H^{-1}F'x\|_H^2 + \frac{1}{2}\|x\|^2_{Y-FH^{-1}F'}, \tag{7}$$

where $\|\xi\|_M^2 = \xi'M\xi$ and $\|\xi\|^2 = \xi'\xi$ for any vector $\xi \in \mathbb{R}^s$ and symmetric matrix $M \in \mathbb{R}^{s\times s}$. We show that $Y - FH^{-1}F' \succ 0$ in the Appendix.

Problem (3) may not have a solution for all $x \in \mathbb{X}$. Let $\mathcal{X} \subseteq \mathbb{X}$ be the set of initial conditions x such that (3) has a solution. For any $x \in \mathcal{X}$, let $U^\star(x)$ and $u^\star(x)$ refer to the optimal solution to (3) and its first element, respectively. Denote the corresponding optimal value of (4) by $V^\star(x)$ and recall this is equal to the optimal value of (3). Using (6) we may express $V^\star(x)$ as

$$V^\star(x) = \frac{1}{2}x'Yx + U^{\star\prime}(x)F'x + \frac{1}{2}U^{\star\prime}(x)HU^\star(x).$$

Under the assumptions stated so far, $\mathcal{X}$ is convex, the functions $U^\star : \mathcal{X} \to \mathbb{U}^N$ and $u^\star : \mathcal{X} \to \mathbb{U}$ are continuous and piecewise affine, and $V^\star : \mathcal{X} \to \mathbb{R}$ is continuous, convex and piecewise quadratic (Bemporad et al., 2002). Let the symbol x^+ denote the predicted successor state for the controlled system, i.e.,

$$x^+ = Ax + Bu^\star(x). \tag{8}$$

We make the following assumption throughout the paper.

Assumption 1. The optimal value function $V^\star(x)$ of $\mathbb{P}(x)$ is a Lyapunov function for the closed-loop system (8), i.e., there exist strictly positive constants a_1, a_2 and a_3 such that $x \in \mathcal{X}$ implies

$$a_1\|x\|_2^2 \le V^\star(x) \le a_2\|x\|_2^2 \tag{9}$$

$$V^\star(x^+) - V^\star(x) \le -a_3\|x\|_2^2. \tag{10}$$

This assumption is guaranteed to hold if the terminal constraint set $\mathbb{X}_f$ and cost functions $\ell(\cdot)$, $V_f(\cdot)$ satisfy control invariance conditions (Mayne, Rawlings, Rao, & Scokaert, 2000). Note that nominal exponential stability of the origin of the closed-loop system follows from Assumption 1.

2.1. Notation and preliminaries

Let $\mathcal{Q} = \{1, \ldots, q\}$ denote the index set of the constraints of (4). For any matrix $M \in \mathbb{R}^{q\times t}$, let M^i and $M^{\mathcal{W}}$ be the row vector and submatrix of row vectors indicated by $i \in \mathcal{Q}$ and the ordered subset $\mathcal{W} \subseteq \mathcal{Q}$, respectively. The ith constraint in (4) is called *inactive* at the optimum, if $G^iU^\star(x) < w^i + E^ix$, and *active* if $G^iU^\star(x) = w^i + E^ix$, where $U^\star(x)$ is the optimal solution to (4) at state x. We say a constraint $i \in \mathcal{Q}$ is *a priori known to be inactive* for a particular $x \in \mathcal{X}$ if we know $G^iU^\star(x) < w^i + E^ix$ before having solved (4) for $U^\star(x)$.

Recall that a quadratic form with symmetric positive definite matrix M defines an ellipsoid $\{\xi \in \mathbb{R}^s | \xi'M\xi \le 1\} \subset \mathbb{R}^s$ centered at the origin and, for any $\xi_0 \in \mathbb{R}^s$, an ellipsoid

$$\{\xi \in \mathbb{R}^s | (\xi - \xi_0)'M(\xi - \xi_0) \le 1\} \tag{11}$$

centered at ξ_0. Let $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the smallest and largest eigenvalue of M, respectively. We state an important property of ellipsoids in the following lemma.

Lemma 2. *Let $M \in \mathbb{R}^{s\times s}$ be a symmetric positive definite matrix, and consider any $\xi \in \mathbb{R}^s$ and $\alpha \ge 0$. Then*

$$\xi'M\xi \le \alpha^2 \quad \text{implies } \|\xi\| \le \frac{\alpha}{\sqrt{\lambda_{\min}(M)}}.$$

Proof. For any symmetric $M \in \mathbb{R}^{s\times s}$

$$\lambda_{\min}(M)\xi'\xi \le \xi'M\xi$$

for all $\xi \in \mathbb{R}^s$ (Bernstein, 2009, Lemma 8.4.3). Therefore $\xi'M\xi \le \alpha^2$ implies $\lambda_{\min}(M)\xi'\xi \le \alpha^2$. Since M is positive definite by assumption, $\lambda_{\min}(M) > 0$ and the claim follows. ■

3. Reduced equivalent MPC problem

Assume the optimal control problem (4) has been solved for the current initial condition x, and hence the optimal sequence of controls $U^\star(x)$ has been determined. The predicted successor state x^+ of the controlled system is given by (8). In the next time step we need to solve (4) for x^+ to find $U^\star(x^+)$. The present section explains how to simplify (4) by removing constraints that can be shown to be inactive for x^+ before actually solving (4) for x^+.