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Abel lemma-based finite-sum inequality and its application to stability analysis for linear discrete time-delay systems☆

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ABSTRACT

This paper is concerned with stability of linear discrete time-delay systems. Note that a tighter estimation on a finite-sum term appearing in the forward difference of some Lyapunov functional leads to a less conservative delay-dependent stability criterion. By using Abel lemma, a novel finite-sum inequality is established, which can provide a *tighter* estimation than the ones in the literature for the finite-sum term. Applying this Abel lemma-based finite-sum inequality, a stability criterion for linear discrete time-delay systems is derived. It is shown through numerical examples that the stability criterion can provide a larger admissible maximum upper bound than stability criteria using a Jensen-type inequality approach and a free-weighting matrix approach.

1. Introduction

Consider the linear discrete time-delay system described by

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d) \\ x(\theta) = \phi(\theta), \quad \theta = -d, -d+1, \dots, 0 \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state; A and A_d are $n \times n$ known real matrices; $\phi(\theta)$ is an initial condition; and the time delay d is a positive integer.

The system (1) is of strong engineering background because a number of practical control systems are implemented through a communication network (Peng, Tian, & Yue, 2011; Xiong & Lam, 2009). As a consequence, during the last decade, study on stability of the system (1) has gained growing attention, and one can refer to Gao and Chen (2007), He, Wu, Liu, and She (2008), Jiang, Han, and Yu (2005), Xu, Lam, and Zhou (2005), Zhang, Xu, and Zou (2008). The objective is to derive a delay-dependent stability criterion such that an admissible maximum upper bound $d_{\max}$ of d can be obtained. The larger $d_{\max}$, the less conservatism of a stability criterion.

Note that a lifting technique can be used to derive a necessary and sufficient condition on stability of the system (1) (Xia, Liu, Shi, Rees, & Thomas, 2007). In fact, introduce an augmented vector as $\Phi(k) := \text{col}\{x(k), x(k-1), \dots, x(k-d)\}$. Then, the first equation of the system (1) is transformed into

$$\Phi(k+1) = \tilde{A}\Phi(k), \quad \tilde{A} := \begin{bmatrix} AE & A_d \\ I & 0 \end{bmatrix} \tag{2}$$

where $E = [I \; 0_{n\times nd}]$. Thus, a necessary and sufficient condition for the stability of the system (1) is that there exists a real matrix $\tilde{P} \in \mathbb{R}^{nd\times nd}$ such that $\tilde{A}^T\tilde{P}\tilde{A} - \tilde{P} < 0$. The drawback of the condition is that the dimensions of the matrices $\tilde{A}$ and $\tilde{P}$ are closely dependent on the delay size d. For a large d, on the one hand, a large number of decision variables are required in seeking a suitable real matrix $\tilde{P}$; and on the other hand, this condition is not easy to use for control synthesis and filter design.

An alternative method for stability analysis of the system (1) is the Lyapunov functional method. One usually chooses a Lyapunov functional candidate as

$$\tilde{V}(k) = x^T(k)Px(k) + \sum_{j=k-d}^{k-1} x^T(j)Qx(j) + d\sum_{j=-d}^{-1}\sum_{i=k+j}^{k-1} \eta^T(i)R\eta(i)$$

where $P > 0$, $Q > 0$ and $R > 0$ are Lyapunov matrices to be determined; and

$$\eta(k) = x(k+1) - x(k). \tag{3}$$

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Taking the forward difference yields

$$\Delta \tilde{V}(k) = x^T(k+1)Px(k+1) - x^T(k)(P-Q)x(k) + d^2\eta^T(k)R\eta(k) - x^T(k-d)Qx(k-d) - d\sum_{j=k-d}^{k-1}\eta^T(j)R\eta(j). \quad (4)$$

A sufficient condition for asymptotic stability of the system (1) is that there exist real matrices $P > 0$, $R > 0$ and $Q > 0$ and a scalar $\varepsilon > 0$ such that $\Delta\tilde{V}(k) \le -\varepsilon x^T(k)x(k) < 0$ for $x(k) \ne 0$. In order to derive a linear matrix inequality (LMI)-based stability criterion, a *challenging* problem is how to estimate the finite-sum term

$$\mathscr{F}(k) := \sum_{j=k-d}^{k-1}\eta^T(j)R\eta(j). \quad (5)$$

Clearly, a tighter lower bound for the term $\mathscr{F}(k)$ definitely contributes to a less conservative stability criterion. Up to date, there are mainly two approaches to deal with $\mathscr{F}(k)$.

(i) *Jensen-type inequality approach*. Applying the Jensen-type inequality, one obtains a lower bound for $\mathscr{F}(k)$, which is given by Jiang et al. (2005)

$$\mathscr{F}(k) \ge \mathscr{B}_1 := \frac{1}{d}[x(k)-x(k-d)]^T R[x(k)-x(k-d)]. \quad (6)$$

(ii) *Free-weighting matrix approach*. In He et al. (2008), a free-weighting matrix approach is proposed to deal with the term $\mathscr{F}(k)$. If gaining an insight into the approach in He et al. (2008), one can find that a different bound for the term $\mathscr{F}(k)$ is used, which reads as

$$\mathscr{F}(k) \ge \mathscr{B}_2 := 2\rho^T(k)N[x(k)-x(k-d)] - d\rho^T(k)Z\rho(k) \quad (7)$$

where $\rho(k)$ is some real vector and $\begin{bmatrix} Z & N \\ N^T & R\end{bmatrix} \ge 0$.

It seems that the lower bound $\mathscr{B}_2$ in (7) is larger than $\mathscr{B}_1$ in (6) because some free-weighting matrices are introduced in $\mathscr{B}_2$. *Unfortunately,* it can be proven that $\mathscr{B}_1 \ge \mathscr{B}_2$. In fact, employing Lemma 1, which is given at the end of this section, with $\kappa = d$, $M_1 = Z$, $M_2 = R$, $S = -N$, $\alpha = \rho(k)$ and $\beta = x(k)-x(k-d)$ yields $\mathscr{B}_1 \ge \mathscr{B}_2$. Moreover, denote $\mathcal{S} := \left\{(N,Z)\,\middle|\,\begin{bmatrix} Z & N \\ N^T & R\end{bmatrix} \ge 0\right\}$. Then

$$\mathscr{B}_1 = \max_{(N,Z)\in\mathcal{S}}\mathscr{B}_2. \quad (8)$$

In fact, taking $N = \frac{1}{d}R$, $Z = \frac{1}{d^2}R$ gives $\mathscr{B}_2 = \mathscr{B}_1$. Therefore, the free-weighting matrix approach does **not** provide a tighter lower bound than the Jensen-type inequality approach for the term $\mathscr{F}(k)$. In other words, stability criteria employing the free-weighting matrix approach are of the same conservatism as those employing the Jensen-type inequality approach for the system (1). Then, some natural questions arise: Does there exist a *tighter* lower bound than $\mathscr{B}_1$ for the term $\mathscr{F}(k)$? If yes, how to get it? Answering these questions is of significance in theory and in practice, which motivates the current study.

In this paper, we propose a new method to estimate the finite-sum term $\mathscr{F}(k)$. By applying Abel lemma, a novel finite-sum inequality for the term $\mathscr{F}(k)$ is derived, which is given as

$$\mathscr{F}(k) \ge \mathscr{B}_3 := \frac{1}{d}\omega_1^T R\omega_1 + \frac{3(d-1)}{d(d+1)}\omega_2^T R\omega_2 \quad (9)$$

where $\omega_1 := x(k) - x(k-d)$ and $\omega_2 := x(k) + x(k-d) - \frac{2}{d-1}\sum_{j=k-d+1}^{k-1}x(j)$. Clearly, the lower bound $\mathscr{B}_3$ in (9) is larger than $\mathscr{B}_1$ due to the fact that $\frac{3(d-1)}{d(d+1)} > 0$. Hence, $\mathscr{B}_3$ is a *tighter* lower bound than $\mathscr{B}_1$ for $\mathscr{F}(k)$. It is worth pointing out that the inequality (9) is similar to the Wirtinger-based integral inequality (Seuret & Gouaisbaut, 2013). Thus, the inequality (9) can be regarded as the discrete-time version of the Wirtinger-based integral inequality. To proceed with, applying the inequality (9), a new delay-dependent stability criterion for the system (1) is formulated. Numerical examples show that the obtained stability criterion can achieve less conservative results than stability criteria using the Jensen-type inequality approach and the free-weighting matrix approach.

To end this section, we introduce three lemmas.

Lemma 1 (Zhang & Han, 2015). *Let α and β be real column vectors with dimensions of n_1 and n_2, respectively. For given real positive symmetric matrices $M_1 \in \mathbb{R}^{n_1\times n_1}$ and $M_2 \in \mathbb{R}^{n_2\times n_2}$, the following inequality holds for any scalar $\kappa > 0$ and matrix $S \in \mathbb{R}^{n_1\times n_2}$ satisfying $\begin{bmatrix} M_1 & S \\ S^T & M_2\end{bmatrix} \ge 0$*

$$-2\alpha^T S\beta \le \kappa\alpha^T M_1\alpha + \kappa^{-1}\beta^T M_2\beta. \quad (10)$$

Lemma 2 (Jiang et al., 2005). *For any constant matrix $R \in \mathbb{R}^{n\times n}$ with $R = R^T > 0$, integers r_1 and r_2 with $r_2 > r_1 > 0$, vector function $w : \{r_1, r_1+1, \ldots, r_2\} \to \mathbb{R}^n$, the following inequality holds*

$$\sum_{j=r_1}^{r_2-1} w^T(j)Rw(j) \ge \frac{1}{r_2-r_1}\left(\sum_{j=r_1}^{r_2-1}w(j)\right)^T R\left(\sum_{j=r_1}^{r_2-1}w(j)\right).$$

Lemma 3 (Abel Lemma (Bromwich, 1959)). *Suppose that $\{f_j\}$ and $\{g_j\}$ are two sequences. Then*

$$\sum_{j=m}^{p} f_j(g_{j+1}-g_j) = (f_{p+1}g_{p+1} - f_m g_m) - \sum_{j=m}^{p} g_{j+1}(f_{j+1}-f_j).$$

2. An Abel lemma-based finite-sum inequality

In this section, we establish an Abel lemma-based finite-sum inequality for the term $\mathscr{S}_R(r_1, r_2)$ given by

$$\mathscr{S}_R(r_1,r_2) := \sum_{j=r_1}^{r_2-1}\eta^T(j)R\eta(j) \quad (11)$$

where r_1 and $r_2 (> r_1)$ are two positive scalars, and η is defined in (3). Let

$$g_j := x(j), \qquad f_j := r_1 + r_2 - 1 - 2j. \quad (12)$$

Then, apply Lemma 3 (Abel lemma) to obtain

$$\begin{aligned}\sum_{j=r_1}^{r_2-1} f_j\eta(j) &= [f_{r_2}x(r_2) - f_{r_1}x(r_1)] + 2\sum_{j=r_1}^{r_2-1}x(j+1) \\ &= (r_1-r_2-1)x(r_2) - (r_2-r_1-1)x(r_1) + 2\sum_{j=r_1+1}^{r_2}x(j) \\ &= -(r_2-r_1-1)\left[x(r_2)+x(r_1) - \frac{2}{r_2-r_1-1}\sum_{j=r_1+1}^{r_2-1}x(j)\right]. \end{aligned} \quad (13)$$

Moreover, it is easy to verify that $\sum_{j=r_1}^{r_2-1} f_j = 0$ and

$$\sum_{j=r_1}^{r_2-1} f_j^2 = \frac{(r_2-r_1)(r_2-r_1-1)(r_2-r_1+1)}{3}. \quad (14)$$

Based on (13) and (14), we now establish a new inequality for the term $\mathscr{S}_R(r_1, r_2)$. In doing so, denote

$$\upsilon_1 := x(r_2) - x(r_1) \quad (15)$$

$$\upsilon_2 := x(r_2) + x(r_1) - \frac{2}{r_2-r_1-1}\sum_{j=r_1+1}^{r_2-1}x(j). \quad (16)$$