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Null controllability of the heat equation using flatness☆

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ABSTRACT

We derive in a direct and rather straightforward way the null controllability of the N-dimensional heat equation in a bounded cylinder with boundary control at one end of the cylinder. We use the so-called *flatness approach*, which consists in parameterizing the solution and the control by the derivatives of a "flat output". This yields an explicit control law achieving the exact steering to zero. Replacing the involved series by partial sums we obtain a simple numerical scheme for which we give explicit error bounds. Numerical experiments demonstrate the relevance of the approach.

1. Introduction

The controllability of the heat equation was first considered in the 1-D case (Fattorini & Russell, 1971; Jones, 1977; Littman, 1978; Luxemburg & Korevaar, 1971) and very precise results were obtained by the classical moment approach. Next using Carleman estimates and duality arguments the null controllability was proved in Fursikov and Imanuvilov (1996) and Lebeau and Robbiano (1995) for any bounded domain in $\mathbb{R}^N$, any control time T, and any control region. This Carleman approach proves very efficient also with semilinear parabolic equations (Fursikov & Imanuvilov, 1996). By contrast the interest for the numerical investigation of the null controllability of the heat equation (or of parabolic equations) is fairly recent: apart from Carthel, Glowinski, and Lions (1994), the first significant contributions are Belgacem and Kaber (2011), Boyer, Hubert, and Le Rousseau (2011), Fernández-Cara and Münch (2011), Fernández-Cara and Münch (2012), Fernández-Cara and Münch (2013), Labbé and Trélat (2006), Münch and Pedregal (2014), Münch and Zuazua (2010), Zheng (2008) and Zuazua (2006); see also Garcia, Osses, and Tapia (2013) for an application to some inverse problems. All the above results rely on some observability inequalities for the adjoint system. A direct approach which does not involve the adjoint problem was proposed in Jones (1977), Lin Guo and Littman (1995), Littman (1978) and Littman and Taylor (2007). In Jones (1977) a fundamental solution for the heat equation with compact support in time was introduced and used to prove null controllability. The results in Jones (1977) and Rosier (2002) can be used to derive control results on a bounded interval with two or one boundary control in some Gevrey class, or on a bounded domain of $\mathbb{R}^N$ with a control supported on the whole boundary (see also Littman & Taylor, 2007). An extension of those results to the semilinear heat equation in 1-D was obtained in Lin Guo and Littman (1995) in a more explicit way through the resolution of an ill-posed problem with data of Gevrey order 2 in t.

In this paper, which builds on the preliminary versions (Martin, Rosier, & Rouchon, 2013a,b), we derive in a straightforward way the null controllability of the heat equation in a bounded cylinder $\Omega = \omega \times (0, 1) \subset \mathbb{R}^N$ with Neumann boundary control on $\omega \times \{1\}$. More precisely given any final time $T > 0$ and initial state $\theta_0 \in L^2(\Omega)$ we provide an explicit and very regular control such that the state reached at time T is exactly zero. We use the so-called *flatness* approach (Fliess, Lévine, Martin, & Rouchon, 1995), which consists in parameterizing the solution θ and the control u by the derivatives of a "flat output" y; this notion was initially introduced for finite-dimensional (nonlinear) systems, and later extended to (in particular) parabolic 1-dimensional PDEs (Laroche, Martin, &

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Rouchon, 2000; Lynch & Rudolph, 2002; Meurer, 2011; Meurer & Zeitz, 2008). Choosing a suitable trajectory for this flat output y then yields an explicit series for a control achieving the exact steering to zero. Note this paper is probably the first example of using flatness for the motion planning of a "truly" N-dimensional PDE. Comparing our results to Lin Guo and Littman (1995) and Littman and Taylor (2007) we note that: (i) our control is not supported on the whole boundary even in dimension $N > 1$; (ii) the control and the solution are Gevrey of order $s \in (1, 2)$ in time; (iii) the control and the solution are developed in series whose easily computed partial sums yield accurate numerical approximations of both the control and the solution.

The paper runs as follows. In Section 3 we consider the control problem in dimension $N = 1$. In Proposition 1 we investigate an ill-posed problem with Cauchy data in a Gevrey class and prove its (global) well-posedness. Theorem 3 then establishes the null controllability in small time for any initial data in L^2. In Section 4 we extend these results to the cylinder $\Omega = \omega\times(0, 1) \subset \mathbb{R}^N$. Section 5 provides accurate error estimates when the various series involved are replaced by their partial sums. Finally in Section 6 some numerical experiments demonstrate the interest of the approach.

2. Preliminaries (Gevrey functions)

In the sequel we consider series with infinitely many derivatives of functions. The notion of Gevrey order is a way of estimating the growth of these derivatives: we say that a function $y \in C^\infty([0, T])$ is *Gevrey of order* $s \geq 0$ *on* $[0, T]$ if there exist positive constants M, R such that

$$\left|y^{(p)}(t)\right| \leq M\frac{p!^s}{R^p} \quad \forall t \in [0, T],\ \forall p \geq 0.$$

More generally if $K \subset \mathbb{R}^N$ is a compact set and y is a function of class C^∞ on K (i.e. y is the restriction to K of a function of class C^∞ on some open neighborhood Ω of K), we say y is *Gevrey of order* s_1 *in* x_1, s_2 *in* $x_2, \ldots, s_N$ *in* x_N *on* K if there exist positive constants $M, R_1, \ldots, R_N$ such that $\forall x \in K,\ \forall p \in \mathbb{N}^N$

$$\left|\partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_N}^{p_N} y(x)\right| \leq M\frac{\prod_{i=1}^{N}(p_i!)^{s_i}}{\prod_{i=1}^{N}R_i^{p_i}}.$$

By definition, a Gevrey function of order s is also of order r for $r \geq s$. Gevrey functions of order 1 are analytic (entire if $s < 1$). Gevrey functions of order $s > 1$ may have a divergent Taylor expansion; the larger s, the "more divergent" the Taylor expansion. Important properties of analytic functions generalize to Gevrey functions of order $s > 1$: the scaling, addition, multiplication and derivation of Gevrey functions of order $s > 1$ is of order s, see Ramis (1978), Rudin (1987) and Yamanaka (1989). But contrary to analytic functions, Gevrey functions of order $s > 1$ may be constant on an open set without being constant everywhere. For example the "step function"

$$\phi_s(t) := \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \\ \dfrac{e^{-(1-t)^{-k}}}{e^{-(1-t)^{-k}} + e^{-t^{-k}}} & \text{if } t \in (0, 1), \end{cases}$$

where $k = (s-1)^{-1}$ is Gevrey of order s on $[0, 1]$ (and in fact on $\mathbb{R}$); notice $\phi_s(0) = 1$, $\phi_s(1) = 0$ and $\phi_s^{(i)}(0) = \phi_s^{(i)}(1) = 0$ for all $i \geq 1$.

In conjunction with growth estimates we will repeatedly use Stirling's formula $n! \sim (n/e)^n\sqrt{2\pi n}$.

3. The one-dimensional heat equation

For simplicity we first study the 1-D heat equation with Neumann boundary control

$$\partial_t\theta(t, x) - \partial_x^2\theta(t, x) = 0, \quad (t, x) \in (0, T) \times (0, 1) \tag{1}$$

$$\partial_x\theta(t, 0) = 0, \quad t \in (0, T) \tag{2}$$

$$\partial_x\theta(t, 1) = u(t), \quad t \in (0, T) \tag{3}$$

with initial condition in $L^2(0, 1)$

$$\theta(0, x) = \theta_0(x), \quad x \in (0, 1).$$

We claim the system (1)–(3) is "flat" with $y(t) := \theta(t, 0)$ as a flat output, meaning there is (in appropriate spaces of smooth functions) a 1–1 correspondence between arbitrary functions $t \mapsto y(t)$ and solutions of (1)–(3).

We first seek a formal solution in the form

$$\theta(t, x) := \sum_{i\geq 0}\frac{x^i}{i!}a_i(t)$$

where the a_i's are functions yet to define. Plugging this expression into (1) yields

$$\sum_{i\geq 0}\frac{x^i}{i!}[a_{i+2} - a_i'] = 0,$$

hence $a_{i+2} = a_i'$ for all $i \geq 0$. On the other hand $y(t) = \theta(t, 0) = a_0(t)$, and (2) implies $a_1(t) = 0$. As a consequence $a_{2i} = y^{(i)}$ and $a_{2i+1} = 0$ for all $i \geq 0$. The formal solution thus reads

$$\theta(t, x) = \sum_{i\geq 0}\frac{x^{2i}}{(2i)!}y^{(i)}(t) \tag{4}$$

while the formal control is given by

$$u(t) = \theta_x(t, 1) = \sum_{i\geq 1}\frac{y^{(i)}(t)}{(2i-1)!}. \tag{5}$$

We now give a meaning to this formal solution by restricting $t \mapsto y(t)$ to be Gevrey of order $s \in [0, 2)$.

Proposition 1. *Let* $s \in [0, 2)$, $-\infty < t_1 < t_2 < \infty$, *and* $y \in C^\infty([t_1, t_2])$ *satisfying for some constants* $M, R > 0$

$$\left|y^{(i)}(t)\right| \leq M\frac{i!^s}{R^i}, \quad \forall i \geq 0,\ \forall t \in [t_1, t_2]. \tag{6}$$

Then the function θ *defined by* (4) *is Gevrey of order* s *in* t *and* $s/2$ *in* x *on* $[t_1, t_2] \times [0, 1]$; *hence the control* u *defined by* (5) *is also Gevrey of order* s *on* $[t_1, t_2]$.

Proof. We must prove the formal series

$$\partial_t^m\partial_x^n\theta(t, x) = \sum_{2i\geq n}\frac{x^{2i-n}}{(2i-n)!}y^{(i+m)}(t) \tag{7}$$

is uniformly convergent on $[t_1, t_2] \times [0, 1]$ with growth estimates of the form

$$\left|\partial_t^m\partial_x^n\theta(t, x)\right| \leq C\frac{m!^s}{R_1^m}\frac{n!^{\frac{s}{2}}}{R_2^n}. \tag{8}$$

By (6), we have for all $(t, x) \in [t_1, t_2] \times [0, 1]$

$$\begin{aligned}\left|\frac{x^{2i-n}}{(2i-n)!}y^{(i+m)}(t)\right| &\leq \frac{M}{R^{i+m}}\frac{(i+m)!^s}{(2i-n)!} \\ &\leq \frac{M}{R^{i+m}}\frac{(2^{i+m}i!\,m!)^s}{(2i-n)!}\end{aligned}$$