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Averaged control[☆]

Enrique Zuazua [1]

BCAM - Basque Center for Applied Mathematics, Mazarredo 14, 48009 Bilbao, Basque Country, Spain
Ikerbasque, Alameda Urquijo 36-5, Plaza Bizkaia, 48011, Bilbao, Basque Country, Spain

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ABSTRACT

We analyze the problem of controlling parameter-dependent systems. We introduce the notion of averaged control according to which the quantity of interest is the average of the states with respect to the parameter.

First we consider the problem of controllability for linear finite-dimensional systems and show that a necessary and sufficient condition for averaged controllability is an averaged rank condition, in the spirit of the classical rank condition for linear control systems, but involving averaged momenta of any order of the matrices generating the dynamics and representing the control action.

We also describe some open problems and directions of possible research, in particular on the average controllability of evolution partial differential equations. In this context we analyze also the averaged version of a classical optimal control problem for a parameter dependent elliptic equation and derive the corresponding optimality system.

1. Introduction

We analyze the problem of controlling systems submitted to parametrized perturbations, either finite or infinite dimensional ones, i.e. ordinary or partial differential equations (ODE or PDE), depending on unknown parameters in a deterministic manner. We look for a control, independent of the values of these parameters, that needs to be designed to perform well, in an averaged sense to be made precise. We do it under an averaged criterion, considering two particularly relevant cases:

- *Parameter dependent ODEs*: We introduce and analyze the problem of controlling the expected or averaged value of the systems states starting from a given and known initial datum and by means of a single control, independent of the parameters involved. First, using classical duality theory, we show that the problem is equivalent to an averaged observability inequality for the adjoint system whose distinguished feature is that all the components take the same final state independently of the value of the parameter. Then, we characterize the averaged controllability property through a suitable rank condition involving the averaged momenta of any order of the operators generating the dynamics and the control ones.
- *Parameter dependent PDEs*: We introduce the same notion of averaged control for parameter-dependent partial differential equations (PDE). By duality this leads to averaged observability problems. As we shall see, this is a challenging topic in which plenty is to be done, requiring significant further work. We also consider the problem of the optimal control for a parameter-dependent family of elliptic PDE. We provide a complete characterization of the optimal control through the corresponding optimality system.

As we shall see, the notion of averaged control addressed in this paper is weaker than the classical one of simultaneous control introduced in Lions (1988, Chapter V).

In this paper we discuss the case in which the parameter-dependence is of deterministic nature. Similar results would hold when the operators involved vary with respect to uncertain parameters with a given probability density and one aims at controlling the expected value of the state. But, to simplify the presentation, we shall focus on the deterministic setting.

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E-mail address: zuazua@bcamath.org.

1 Tel.: +34 688806887; fax: +34 946 567 843.

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Averaged controllability is equivalent, by duality, to a property of averaged observability in which the goal is to estimate the norm of the data of the parameter-dependent adjoint system, out of partial measurements done on the averages with respect to the unknown parameters. This property is of interest on its own, when dealing with the observability of parameter-dependent systems. The actual realization of the system depending on the parameter being unknown, it is natural to address the problem based on the measurements done on averages.

The notion of averaged controllability, as formulated here, has not been analyzed until now. The nature of the results and open problems arising in its study both for finite-dimensional and infinite-dimensional systems are a good evidence of its relevance and suitability. As we shall see, when facing parameter-dependent situations, the averaged control is a natural first guess. Our results not only allow establishing whether a system is controllable in an averaged sense, but also to derive characterizations that turn out to be algorithmic and serve for computational purposes.

The problem of averaged control is also related to the issue of robust control, that has been reversively addressed in the literature from different viewpoints. The interested reader is referred to the book (Ackermann et al., 2002), for instance, and to Masterkov and Rodina (2007) where output non-anticipating feedback optimal control results are derived for linear uncertain finite-dynamical systems (see also Lu & Zuazua, 2014 for the first results on non-anticipating control of heat-like equations). We also refer to Petersen (2009) and Savkin and Petersen (1996) for the analysis of the problem of possible controllability of uncertain systems, to Johnson and Nerurkar (1998) and references therein for the problem of controlling finite-dimensional systems subject to some random non-autonomous dynamics and to Seo, Chung, Park, and Lee (2005) for related notions of robust control.

The rest of this paper is organized as follows. In Section 2 we discuss the finite-dimensional averaged controllability problem. The equivalence with the property of averaged observability and with an averaged rank condition is proved. In Section 3 we discuss the comparison of these problems and results with the existing ones on simultaneous control. In Section 4.3 we discuss the PDE case, formulating several model problems and analyzing in detail the problem of elliptic averaged optimal control. We close this paper pointing towards some open problems and future directions of research.

2. Finite dimensional linear systems

2.1. Problem formulation and main result

Consider the finite dimensional linear control system

$$\begin{cases} x'(t) = A(\nu)x(t) + B(\nu)u(t), \quad 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{1}$$

In (1) the (column) vector valued function $x(t,\nu) = \big(x_1(t,\nu), \dots, x_N(t,\nu)\big) \in \mathbf{R}^N$ is the state of the system, $A(\nu)$ is an $N \times N$-matrix governing its free dynamics and $u = u(t)$ is an M-component control vector in $\mathbf{R}^M$, $M \leq N$, entering and acting on the system through the control operator $B(\nu)$, an $N \times M$ parameter-dependent matrix.

The matrices A and B are assumed to depend on a parameter ν in a measurable manner, although our analysis would also be valid for the multi-parameter case. To simplify the notation we will assume that $\nu \in \mathbf{R}$, although a similar analysis can be developed when ν is a multivalued parameter or even a random one, living in a probability space. To fix ideas we will assume that the parameter ν ranges within the interval $(0, 1)$. We also assume that A and B are uniformly bounded with respect to ν, so as to ensure (by Lebesgue dominated convergence theorem) the integrability of the solutions of (1) (and the corresponding adjoint system) with respect to ν.

Note however that the initial datum $x^0 \in \mathbf{R}^N$ to be controlled, in principle, is independent of the parameter ν. But the state of the system itself $x(t,\nu)$ depends on ν. The case where x^0 depends on ν will be discussed as well.

The motivation of the problem we consider is the following: We address the controllability of this system whose initial datum is given, known and fully determined. However, the dynamics of the state is governed by a parametrized operator $A(\nu)$, the same as the control operator $B(\nu)$. The effective value of the parameter ν being unknown, we aim at choosing a control that would perform optimally in an averaged sense, i.e. so that, rather than controlling specific realizations of the state, the average with respect to ν is controlled. This allows building a control independent of the parameter and making a robust compromise of all the possible realizations of the system for the various possible values of the unknown parameter ν.

More precisely, the problem can be formulated as follows: *Given a control time $T > 0$ and arbitrary initial data x^0 and final target $x^1 \in \mathbf{R}^N$ we look for a control u such that the solution of (1) satisfies*

$$\int_0^1 x(T,\nu)d\nu = x^1. \tag{2}$$

Note that, contrarily to the case in which A and B are independent of ν, we cannot reduce the problem to the particular case where $x^1 \equiv 0$. This will also be observed at the level of the dual observability problem. Thus, the final (2) needs to be considered for all possible targets x^1.

Note also that this concept of averaged controllability differs and is weaker from that of simultaneous controllability in which one is interested on controlling all states simultaneously and not only its average.

The particular case where A is independent of ν but $B = B(\nu)$ can be handled quite easily. Indeed, consider the system

$$\begin{cases} x'(t) = Ax(t) + B(\nu)u(t), \quad 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{3}$$

Obviously, the state $x = x(t,\nu)$ depends on ν and the notion of averaged controllability property (2) makes sense. But in the present case the problem is easy to solve since

$$y(t) = \int_0^1 x(t,\nu)d\nu$$

solves the system

$$\begin{cases} y'(t) = Ay(t) + \hat{B}u(t), \quad 0 < t < T, \\ y(0) = x^0, \end{cases} \tag{4}$$

where $\hat{B} = \int_0^1 B(\nu)d\nu$ is the average of the control operators.

Accordingly, when $A(\nu) = A$ for all ν, the averaged controllability property holds if and only if the pair $(A, \hat{B} = \int_0^1 B(\nu)d\nu)$ satisfies the rank condition:

$$\operatorname{rank}\Big[A^j \int_0^1 B(\nu)d\nu : 0 \leq j \leq N-1\Big] = N. \tag{5}$$

But this averaging principle does not apply when the operators A depend on ν. In this more general setting, the property of averaged controllability will be characterized through a rank condition in the same spirit, but involving the averages of A and all its powers, together with the control operator B, with respect to ν. However, as we shall see, in general, contrarily to the case in which A is independent of ν (see Lee & Markus, 1967 and Trélat, 2005), this condition will involve powers of arbitrary order, and not only a finite number of powers up to order $N-1$ as in (5).