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Brief paper

On stability and stabilization of periodic discrete-time systems with an application to satellite attitude control[☆]

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ABSTRACT

An alternative stability analysis theorem for nonlinear periodic discrete-time systems is presented. The developed theorem offers a trade-off between conservatism and complexity of the corresponding stability test. In addition, it yields a tractable stabilizing controller synthesis method for linear periodic discrete-time systems subject to polytopic state and input constraints. It is proven that in this setting, the proposed synthesis method is strictly less conservative than available tractable synthesis methods. The application of the derived method to the satellite attitude control problem results in a large region of attraction.

1. Introduction

This work deals with stability and stabilization of periodically time-varying systems, or shortly, *periodic systems*. Stability analysis and stabilization of periodic systems are typically handled by means of periodically time-varying standard Lyapunov functions (LFs), see Jiang and Wang (2002) for the nonlinear case and Bittanti and Colaneri (2009) for the linear case. For most of the available controller synthesis methods for periodic systems, existence of a periodically time-varying LF for the closed-loop dynamics can be derived, either directly or by the converse result in Jiang and Wang (2002). Consider methods based on the periodic Riccati equation Bittanti, Colaneri, and De Nicolao (1991) and Varga (2008), output feedback schemes De Souza and Trofino (2000), $\mathcal{H}_2$ synthesis for the case of linear periodic systems with polytopic uncertainties Farges, Peaucelle, Arzelier, and Daafouz (2007), eigenvalue assignment Brunovský (1970), Kabamba (1986) controllability Longhi and Zulli (1995), model predictive control Böhm (2011), Gondhalekar and Jones (2011), and control with saturation Zhou, Zheng, and Duan (2011). In the monograph (Bittanti & Colaneri, 2009, Chapter 13), a thorough exposition of existing results on stabilization techniques, including also frequency domain considerations or lifting techniques, is presented.

In the presence of constraints, however, stability analysis based on periodically time-varying standard LFs can yield a conservative region of attraction, as shown recently in Böhm, Lazar, and Allgower (2012). Therein, a relaxed stability analysis theorem was derived for autonomous nonlinear periodic systems. The main idea behind this relaxation is that the Lyapunov function is not required to decrease at each time instant, as in Bittanti and Colaneri (2009) or in Jiang and Wang (2002) for the linear case, but at each period. This paper considers stabilization of linear periodic systems with inputs, subject to polytopic state and input constraints, by means of linear periodic state-feedback control laws. The presence of input constraints further motivates the need for a relaxation of the classical stability analysis theorems Bittanti and Colaneri (2009) and Jiang and Wang (2002). For the case of periodic systems with inputs, however, the relaxed periodic Lyapunov conditions in Böhm et al. (2012) lead to a nonlinear and non-convex optimization problem which is not tractable.

Motivated by the current status, we propose an alternative stability analysis theorem for nonlinear periodic systems. This new result allows the establishment of a tractable constrained synthesis

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method for linear periodic systems, by choosing quadratic periodic Lyapunov functions. We show how the constrained synthesis problem with linear periodic state-feedback can be solved by decomposing the original non-convex optimization problem in a finite set of semi-definite optimization problems having linear matrix inequalities (LMIs) as constraints. The equivalence between the original non-convex problem and the set of semi-definite optimization problems is formally proven. The method is applied successfully in the challenging magnetic satellite attitude control problem. The developed synthesis method yields a large region of attraction for the resulting closed-loop system while providing non-trivial performance guarantees.

The remaining part of this paper is structured as follows. Existing results on Lyapunov stability for periodic systems are briefly discussed in Section 2. The problem formulation as well as solutions from existing approaches are presented in Section 3. The main results are established in Section 4. Application of the established results to the satellite attitude control problem is presented in Section 5, while conclusions are drawn in Section 6.

Notation and basic definitions: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{x \in \Pi \mid x \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For $N \in \mathbb{Z}_{\geq 1}$, $\Pi^N := \Pi \times \cdots \times \Pi$. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the ith element of x and $\|x\|$ denotes its 2-norm, i.e., $\|x\| := \sqrt{\sum_{i=1}^n |[x]_i|^2}$. The transpose of a matrix $X \in \mathbb{R}^{n\times m}$ is denoted by $X^\top$. For a symmetric matrix $Z \in \mathbb{R}^{n\times n}$ let $Z \succ 0 (\succeq 0)$ denote that Z is positive definite (semi-definite). For a positive definite matrix $Z \in \mathbb{R}^{n\times n}$ let $\lambda_{\min(\max)}(Z)$ denote its smallest (largest) eigenvalue. Moreover, for a block symmetric matrix $Z = \begin{bmatrix} a & b^\top \\ b & c \end{bmatrix}$, where a, b, c are matrices of appropriate dimensions, the symbol $\star$ is used to denote the symmetric part, i.e., $\begin{bmatrix} a & \star \\ b & c \end{bmatrix} = \begin{bmatrix} a & b^\top \\ b & c \end{bmatrix}$. For the definition of functions of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$, refer to Böhm et al. (2012).

2. Preliminaries

Let $n, m \in \mathbb{Z}_+$ be integers and let $\mathbb{X} : \mathbb{Z}_+ \to \mathbb{R}^n$ and $\mathbb{U} : \mathbb{Z}_+ \to \mathbb{R}^m$ be maps that assign to each $k \in \mathbb{Z}_+$ a subset of $\mathbb{R}^n$ and a subset of $\mathbb{R}^m$ respectively, which contain the origin in their interior. We consider time-varying nonlinear systems of the form

$$x(k+1) = f(k, x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary nonlinear map such that $f(k, 0, 0) = 0$, for all $k \in \mathbb{Z}_+$. The vector $x(k) \in \mathbb{X}(k)$ is the system state at time $k \in \mathbb{Z}_+$ and $u(k) \in \mathbb{U}(k)$ is the system input at time $k \in \mathbb{Z}_+$.

Definition 1. The system (1) is called *periodic* if there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $k \in \mathbb{Z}_+$ it holds (i) $\mathbb{X}(k) = \mathbb{X}(k+N)$; (ii) $\mathbb{U}(k) = \mathbb{U}(k+N)$; (iii) $f(k, x, u) = f(k+N, x, u)$ for all $x \in \mathbb{X}(k)$, for all $u \in \mathbb{U}(k)$. Furthermore, the smallest such $N \in \mathbb{Z}_{\geq 1}$ is called the *period* of system (1).

We consider a periodically time-varying state feedback control law $g : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^m$ such that $g(k, 0) = 0$, for all $k \in \mathbb{Z}_+$, $g(k, x) = g(k+N, x)$, for all $k \in \mathbb{Z}_+$, and $g(k, x(k)) \in \mathbb{U}(k)$, for all $k \in \mathbb{Z}_+$ and for all $x(k) \in \mathbb{X}(k)$. We assume, for simplicity, that the period of the control law is equal to the period of system (1). The corresponding closed-loop system is

$$x(k+1) = f(k, x(k), g(k, x(k))), \quad k \in \mathbb{Z}_+. \tag{2}$$

System (2) is periodic with period N, since $f(k+N, x, g(k+N, x)) = f(k, x, g(k, x))$. In what follows, let $\mathbb{X}_0 := \mathbb{X}(0)$ and define $\overline{\mathbb{X}} := \bigcup_{k=0}^{N-1} \mathbb{X}(k)$. As such, all state trajectories of system (2) with $x(0) \in \mathbb{X}_0$ satisfy $x(k) \in \overline{\mathbb{X}}$, for all $k \in \mathbb{Z}_+$. For clarity of exposition, we will consider constant input and state dimensions for all modes of the periodic system. The classical time-invariant unconstrained state-space and input domain is recovered by setting $\mathbb{X}(k) = \mathbb{R}^n$, $\mathbb{U}(k) = \mathbb{R}^m$, for all $k \in \mathbb{Z}_+$.

We adopt the notions of asymptotic stability in a set $\mathbb{X}_0$ (AS($\mathbb{X}_0$)), exponential stability in a set $\mathbb{X}_0$ (ES($\mathbb{X}_0$)) and region of attraction (ROA) for system (2) from Böhm et al. (2012). Next, the notion of a *periodically positively invariant* (PPI) sequence of sets is recalled. Let $\{\mathbb{D}(\pi)\}_{\pi\in\mathbb{Z}_{[0,N-1]}}$ denote a sequence of sets with $\mathbb{D}(\pi) \subseteq \mathbb{X}(\pi)$ for all $\pi \in \mathbb{Z}_{[0,N-1]}$.

Definition 2. The sequence $\{\mathbb{D}(\pi)\}_{\pi\in\mathbb{Z}_{[0,N-1]}}$ is called periodically positively invariant for system (2) if for each $\pi \in \mathbb{Z}_{[0,N-1]}$, each $k \in \{iN + \pi\}_{i\in\mathbb{Z}_+}$ and $x(k) \in \mathbb{D}(\pi)$, it holds that $x(k+N) \in \mathbb{D}(\pi)$ and $x(k+j) \in \mathbb{X}(k+j)$, for all $j \in \mathbb{Z}_{[1,N-1]}$.

The following stability theorems correspond to Böhm et al. (2012) and Jiang and Wang (2002) respectively. These results are adapted for system (2) and modified appropriately in order to provide a framework compatible with the results established in this article.

Theorem 1 (*Jiang & Wang, 2002*). *Let $\{\mathbb{X}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ be a PPI sequence of sets w.r.t. (2). Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$ and let $x(\cdot)$ be a solution to (2) with $x(0) := \xi \in \mathbb{X}(0)$. Let $V : \mathbb{Z}_+ \times \overline{\mathbb{X}} \to \mathbb{R}_+$ be a function, such that $V(k, x) = V(k+N, x)$, for all $k \in \mathbb{Z}_+$, and moreover, for all $k \in \mathbb{Z}_+$ it holds that*

$$\alpha_1(\|\xi\|) \leq V(k, \xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{X}(k) \tag{3a}$$

$$V(k+1, f(k, x(k), g(k, x(k)))) \leq \rho V(k, x(k)), \quad \forall \xi \in \mathbb{X}(0). \tag{3b}$$

Then, system (2) is AS($\mathbb{X}_0$).

Theorem 2 (*Böhm et al., 2012*). *Let $\{\mathbb{X}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ be a PPI sequence of sets w.r.t. (2). Let $\alpha_1, \alpha_2\ \bar{\alpha}_j$, $j \in \mathbb{Z}_{[1,N-1]}$ be $\mathcal{K}_\infty$ functions, $\eta \in \mathbb{R}_{[0,1)}$ and $x(\cdot)$ be a solution to (2) with $x(0) := \xi \in \mathbb{X}(0)$. Let $V : \mathbb{Z}_+ \times \overline{\mathbb{X}} \to \mathbb{R}_+$ be a function, such that $V(k, x) = V(k+N, x)$, for all $k \in \mathbb{Z}_+$, and moreover, for all $k \in \mathbb{Z}_+$, for all $j \in \mathbb{Z}_{[1,N-1]}$, it holds that*

$$\|x(j)\| \leq \bar{\alpha}_j(\|x(j-1)\|), \quad \forall \xi \in \mathbb{X}(0) \tag{4a}$$

$$\alpha_1(\|\xi\|) \leq V(k, \xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{X}(k) \tag{4b}$$

$$V(k+N, x(k+N)) \leq \eta V(k, x(k)), \quad \forall \xi \in \mathbb{X}(0). \tag{4c}$$

Then, system (2) is AS($\mathbb{X}_0$).

3. Problem formulation

We consider non-autonomous linear periodic systems

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{5}$$

where $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times m}$, and $A(k) = A(k+N)$, $B(k) = B(k+N)$, for all $k \in \mathbb{Z}_+$. Equivalently to the nonlinear case, by choosing a linear periodic state-feedback control law with period N, i.e.,

$$u(k) = g(k, x(k)) := K(k)x(k), \tag{6}$$

with $K(k) = K(k+N)$, the closed-loop system is

$$x(k+1) = (A(k) + B(k)K(k))x(k). \tag{7}$$

Next, we consider that system (5) is subject to polytopic state periodic constraints

$$\mathbb{X}(k) := \{x \in \mathbb{R}^n : c_i(k)x \leq 1,\ \forall (i, k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+\}, \tag{8}$$