Automatica 50 (2014) 3209–3215

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate ☆

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ARTICLE INFO

Article history:
Received 8 April 2013
Received in revised form
19 May 2014
Accepted 15 July 2014
Available online 4 November 2014

Keywords:
Multi-agent systems
Markov models
Equilibrium
Eigenvalues

ABSTRACT

The process by which new ideas, innovations, and behaviors spread through a large social network can be thought of as a networked interaction game: Each agent obtains information from certain number of agents in his friendship neighborhood, and adapts his idea or behavior to increase his benefit. In this paper, we are interested in how opinions, about a certain topic, form in social networks. We model opinions as continuous scalars ranging from 0 to 1 with 1 (0) representing extremely positive (negative) opinion. Each agent has an initial opinion and incurs some cost depending on the opinions of his neighbors, his initial opinion, and his stubbornness about his initial opinion. Agents iteratively update their opinions based on their own initial opinions and observing the opinions of their neighbors. The iterative update of an agent can be viewed as a myopic cost-minimization response (i.e., the so-called best response) to the others' actions. We study whether an equilibrium can emerge as a result of such local interactions and how such equilibrium possibly depends on the network structure, initial opinions of the agents, and the location of stubborn agents and the extent of their stubbornness. We also study the convergence speed to such equilibrium and characterize the convergence time as a function of aforementioned factors. We also discuss the implications of such results in a few well-known graphs such as Erdos–Renyi random graphs and small-world graphs.

1. Introduction

Rapid expansion of online social networks, such as friendships and information networks, in recent years has raised an interesting question: how do opinions form in a social network? The opinion of each person is influenced by many factors such as his friends, news, political views, and area of professional activity. Understanding such interactions and predicting how specific opinions spread throughout social networks has triggered vast research by economists, sociologist, psychologists, physicists, etc.

The social network can be modeled as a graph where agents are the vertices and edges indicate pairwise acquaintances. There has been an interesting line of research trying to explain emergence of new phenomenon, such as spread of innovations and new technologies, based on local interactions among agents, e.g., Ellison (1993), Montarani and Saberi (2009). Roughly speaking, a coordination game is played between the agents in which adopting a common strategy has a higher payoff and agents behave according to (noisy) best-response dynamics. There is also a rich and still growing literature on social learning using a Bayesian perspective where individuals observe the actions of others and update their beliefs iteratively about an underlying state variable, e.g., Acemoglu, Dahleh, Lobel, and Ozdaglar (2011), Banerjee and Fudenberg (2004). There is also opinion dynamics based on non-Bayesian models, e.g., those in Acemoglu, Ozdaglar, and ParandehGheibi (2010), Borkar, Nair, and Sanketh (2010), DeGroot (1974). As reported in Acemoglu et al. (2010), it is significantly more difficult to analyze social networks with several forceful agents that do not change their opinions and requires a different mathematical approach. Our model is closely related to the non-Bayesian framework, this keeps the computations tractable and can characterize the equilibrium in presence of agents that are biased towards their initial opinions (the so-called partially stubborn agents

☆ This research was supported by AFOSR MURI grant FA 9550-10-1-0573 and ARO MURI grant W911NF-12-1-0385. The material in this paper was partially presented at the 2013 American Control Conference (ACC13), June 17–19, 2013, Washington, DC, USA. This paper was recommended for publication in revised form by Associate Editor Denis Dochain under the direction of Editor Frank Allgöwer.

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http://dx.doi.org/10.1016/j.automatica.2014.10.034
0005-1098/

in our paper) or do not change their opinions at all (the so-called fully stubborn agents in our paper). Furthermore, the equilibrium behavior is relevant only if the convergence time is reasonable (Ellison, 1993). Thus, we develop bounds on the rate of convergence that depend on the structure of the social network (such as the diameter of the graph and the relative degrees of stubborn and non-stubborn agents), and the location of stubborn agents and their levels of stubbornness.

We consider a social graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ consisting of n agents, where agents are the vertices and edges indicate pairwise acquaintances. We model opinions as continuous scalars ranging from 0 to 1 with 1 (0) representing extremely positive (negative) opinion. For example, such scalers could represent people opinions about the economic situation of the country, ranging from 0 to 1, with an opinion 1 corresponding to perfect satisfaction with the current economy and 0 representing an extremely negative view towards the economy. Agents have some private initial opinions and iteratively update their opinions based on observing the opinions of their neighbors and their *stubbornness* with respect to their own initial opinions. We study whether an equilibrium can emerge as a result of such local interactions and how such equilibrium possibly depends on the network structure, initial opinions of the agents, and the location of stubborn agents and the extent of their stubbornness. We also study the convergence speed to such equilibrium and characterize the convergence time as a function of aforementioned factors.

When there are no stubborn agents, our model reduces to a *continuous coordination game* where the (noisy) best-response dynamics converge to *consensus* (i.e., a common opinion in which the impact of each agent is directly proportional to its degree in the social network). In this case, the convergence issues are already well understood in the context of consensus and distributed averaging, e.g., Jadbabaie, Lin, and Morse (2003), Lorenz and Lorenz (2010), Olshevsky and Tsitsiklis (2008), Tsitsiklis, Bertsekas, and Athans (1986). Thus we do not consider this case in this paper.

Main contributions. In this paper, we investigate the convergence issues in presence of stubborn agents. In this case, the opinions do not converge to consensus; however, the opinion of each agent converges to a convex combination of the initial opinions of the stubborn agents. Then our main contributions are the following:

- We exactly characterize the impact of each stubborn agent on such an equilibrium based on appropriately defined hitting probabilities of a random walk over the social network. We also give an interesting electrical network interpretation of the equilibrium.
- Since the exact characterization of convergence time is difficult, we derive appropriate upper-bounds and lower-bounds on the convergence time by extending the frameworks of Diaconis and Stroock (1991) and Sinclair (1992) to approximate *the largest eigenvalue of sub-stochastic matrices*. In particular, we develop a technique based on completing sub-stochastic matrices to stochastic matrices by adding fictitious stubborn nodes to the social graph.

Basic notations. All the vectors are column vectors. x^T denotes the transpose of vector x. A diagonal matrix with elements of vector x as diagonal entries is denoted by diag(x). x_{max} means the maximum element of vector x. Similarly, x_{min} is the minimum element of vector x. $\mathbb{1}_n$ denotes a vector of all ones of size n. $|S|$ denotes the cardinality of set S. Given two functions f and g, $f = O(g)$ if $\sup_n |f(n)/g(n)| < \infty$. $f = \Omega(g)$ if $g = O(f)$. If both $f = O(g)$ and $f = \Omega(g)$, then $f = \Theta(g)$. We will use the following convenient scalar product and its corresponding norm: given vectors z, y, π in $\mathbb{R}^n$, $\langle z, y\rangle_\pi = \sum_{i=1}^n z_i y_i \pi_i$, and $\|z\|_\pi := \left(\sum_{i=1}^n z_i^2 \pi_i\right)^{1/2}$.

2. Model and definitions

Consider a social network with n agents, denoted by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ where agents are the vertices and edges indicate the pairs of agents that have interactions. For each agent i, define its neighborhood ∂_i as the set of agents that node i interacts with, i.e., $\partial_i := \{j : (i,j) \in \mathcal{E}\}$. Each agent i has an initial opinion $x_i(0) \in [0, 1]$. Let $x(0) := [x_1(0) \cdots x_n(0)]^T$ denote the vector of initial opinions. We assume each agent i has a cost function of the form

$$J_i(x_i, x_{\partial_i}) = \frac{1}{2}\sum_{j\in\partial_i}(x_i - x_j)^2 + \frac{1}{2}K_i(x_i - x_i(0))^2, \tag{1}$$

that he tries to minimize where $K_i \geq 0$ measures the *stubbornness* of agent i regarding his initial opinion.[2] When none of the agents are stubborn, correspondingly K_i's are all zero, the above formulation defines a *coordination game* with continuous payoffs because any vector of opinions $x = [x_1 \cdots x_n]^T$ with $x_1 = x_2 = \cdots = x_n$ is a *Nash equilibrium*. Here, we consider a synchronous version of the game between the agents. At each time, every agent observes the opinions of his neighbors and updates his opinion based on these observations and also his own initial opinion in order to minimize his cost function. It is easy to check that, for every agent i, the best-response strategy is

$$x_i(t+1) = \frac{1}{d_i + K_i}\sum_{j\in\partial_i} x_j(t) + \frac{K_i}{d_i + K_i}x_i(0), \tag{2}$$

where $d_i = |\partial_i|$ is the degree of node i in graph $\mathcal{G}$. Similar models have been considered in social influence theory, e.g., see Friedkin and Johnsen (1999) where the model assessment is also done by comparing the observed and predicted opinions of groups. Define a matrix $A_{n\times n}$ such that $A_{ij} = \frac{1}{d_i+K_i}$ for $(i,j) \in \mathcal{E}$ and zero otherwise. Also define a diagonal matrix $B_{n\times n}$ with $B_{ii} = \frac{K_i}{d_i+K_i}$ for $1 \leq i \leq n$. Thus, in the matrix form, the best response dynamics are given by

$$x(t+1) = Ax(t) + Bx(0). \tag{3}$$

Iterating (3) shows that the vector of opinions at each time $t \geq 0$ is

$$x(t) = A^t x(0) + \sum_{s=0}^{t-1} A^s B x(0). \tag{4}$$

In the rest of the paper, we investigate the existence of equilibrium, $x(\infty) := \lim_{t\to\infty} x(t)$, under the dynamics (3) in different social networks, with stubborn agents. The equilibrium behavior is relevant only if the convergence time is reasonable (Ellison, 1993). Thus we also characterize the convergence time of the dynamics, i.e., the amount of time that it takes for the agents' opinions to get close to the equilibrium. To be specific, we investigate the convergence issues under the following assumption.

Assumption 1. (i) $\mathcal{G}$ is an undirected connected graph (otherwise, we can consider opinion dynamics separately over each connected subgraph). (ii) At least one agent is stubborn, i.e., $K_i > 0$ for at least one $i \in \mathcal{V}$ (otherwise, it is well known that the dynamics in (2) converge to consensus, i.e. $x_i(\infty) = \frac{1}{2|\mathcal{E}|}\sum_{j=1}^n d_j x_j(0)$ for all i).

[2] Although we have considered uniform weights for the neighbors, the results in the paper hold under a more general setting when each agent puts a weight w_{ij} for his neighbor j.