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Brief paper

Linearization of control systems: A Lie series approach☆

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ABSTRACT

We address the problem of static state linearization of multi-input nonlinear control systems via coordinate transformation. Necessary and sufficient geometric conditions, in terms of certain set of vector fields associated with the system, were obtained in the early eighties stating the fact that such set of vector fields should be commutative and of constant rank. The state linearization problem, i.e., the finding of linearizing coordinates, was thus reduced to solving a set of partial differential equations. The objective of this paper is to provide an algorithm allowing to compute explicitly the linearizing state coordinates. The algorithm is performed using a maximum of $n-1$ steps (n being the dimension of the system) and is made possible by extending the explicit solvability of the Flow-Box Theorem to a commutative set of vector fields. Examples are provided to illustrate the results. An extension of the method to dynamic feedback linearization is also outlined.

1. Introduction

Let first consider a smooth vector field $\mathbf{v}$ on $\mathbb{R}^n$, or equivalently, its associated dynamical system

$$\Xi_{\mathbf{v}} : \dot{\boldsymbol{x}} = \mathbf{v}(\boldsymbol{x}) \triangleq \begin{cases} \dot{\boldsymbol{x}}_1 = \mathbf{v}_1(\boldsymbol{x}) \\ \dot{\boldsymbol{x}}_2 = \mathbf{v}_2(\boldsymbol{x}) \\ \cdots \\ \dot{\boldsymbol{x}}_n = \mathbf{v}_n(\boldsymbol{x}). \end{cases}$$

Under the change of coordinates $\boldsymbol{z} = \varphi(\boldsymbol{x})$ the system $\Xi_{\mathbf{v}}$ is transformed into

$$\Xi_{\tilde{\mathbf{v}}} : \dot{\boldsymbol{z}} = \tilde{\mathbf{v}}(\boldsymbol{z}) \triangleq \begin{cases} \dot{\boldsymbol{z}}_1 = \tilde{\mathbf{v}}_1(\boldsymbol{z}) \\ \dot{\boldsymbol{z}}_2 = \tilde{\mathbf{v}}_2(\boldsymbol{z}) \\ \cdots \\ \dot{\boldsymbol{z}}_n = \tilde{\mathbf{v}}_n(\boldsymbol{z}) \end{cases}$$

where the vector fields $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are φ-related by the partial differential equation $\tilde{\mathbf{v}}(\varphi(\boldsymbol{x})) = \frac{\partial \varphi}{\partial \boldsymbol{x}}\mathbf{v}(\boldsymbol{x})$. The problem of finding a new coordinate system in which the vector field $\mathbf{v}$ takes its simplest form is centuries old but still of interest nowadays (Arnold, 1988; Belitskii, 1975, 2002; Bruno, 1989; Cigogna & Walcher, 2002; Gaeta, 2002; Pérez-Marco, 2003; Walcher, 2000). A well-known fact is that when the vector field $\mathbf{v}$ is nonsingular at $\boldsymbol{x}_0$, i.e., $\mathbf{v}(\boldsymbol{x}_0) \neq 0$, then there is a change of coordinates $\boldsymbol{z} = \varphi(\boldsymbol{x})$ in which $\Xi_{\tilde{\mathbf{v}}}$ takes its simplest form

$$\Xi_{\tilde{\mathbf{v}}} : \dot{\boldsymbol{z}} = \tilde{\mathbf{v}}(\boldsymbol{z}) \triangleq \begin{cases} \dot{\boldsymbol{z}}_1 = 0 \\ \dot{\boldsymbol{z}}_2 = 0 \\ \cdots \\ \dot{\boldsymbol{z}}_{n-1} = 0 \\ \dot{\boldsymbol{z}}_n = 1. \end{cases}$$

This fact, known as *Straightening* or *Flow-Box Theorem*, generalizes to a family of vector fields as following: take $\Delta(\boldsymbol{x}) = \{\mathbf{v}^1(\boldsymbol{x}), \ldots, \mathbf{v}^m(\boldsymbol{x})\}$ a set of vector fields of constant rank m around a point $\boldsymbol{x}_0$ with commutative vector fields, i.e., $[\mathbf{v}^i(\boldsymbol{x}), \mathbf{v}^j(\boldsymbol{x})] = 0$ for $1 \leq i, j \leq m$. Then new coordinates $\boldsymbol{z} = \varphi(\boldsymbol{x})$ can be found such that $\varphi_* \mathbf{v}^i = \mathbf{e}_i$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. When the vector field $\mathbf{v}$ is singular at $\boldsymbol{x}_0$ ($\mathbf{v}(\boldsymbol{x}_0) = 0$), the notion of linearization and later that of normal form was introduced by Henri Poincaré and followed by a vast literature (see Arnold, 1988; Belitskii, 1975, 2002; Bruno, 1989; Cigogna & Walcher, 2002; Gaeta, 2002; Pérez-Marco, 2003; Walcher, 2000). Poincaré showed that a formal diffeomorphism $\boldsymbol{z} = \varphi(\boldsymbol{x})$ can be found that maps $\mathbf{v}$ into $\tilde{\mathbf{v}}(\boldsymbol{z}) = \mathbf{F}\boldsymbol{z}$ when the spectrum $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $\mathbf{F} = \frac{\partial \mathbf{v}}{\partial \boldsymbol{x}}(\boldsymbol{x}_0)$ is not resonant, i.e., there is no relation of the form

$$\boldsymbol{\alpha}_1\lambda_1 + \cdots + \boldsymbol{\alpha}_n\lambda_n - \lambda_j = 0, \quad \text{for any } 1 \leq j \leq n$$

with $\boldsymbol{\alpha}_i \geq 0$ positive integers such that $\boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_n \geq 2$.

In the late seventies, Krener (1973) adapted Poincaré's method to control systems. Let

$$\Sigma_{(\boldsymbol{f},\mathbf{g})} : \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \mathbf{g}(\boldsymbol{x})u = \boldsymbol{f}(\boldsymbol{x}) + \mathbf{g}_1(\boldsymbol{x})u_1 + \cdots + \mathbf{g}_m(\boldsymbol{x})u_m,$$

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where $\boldsymbol{f}, \mathbf{g}_1, \ldots, \mathbf{g}_m$ are smooth vector fields on $\mathbb{R}^n$, $\boldsymbol{f}(0) = 0$ and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ the control input. Krener considered and solved the following problem (Krener, 1973):

Problem. When does there exists a change of coordinates $\boldsymbol{z} = \varphi(\boldsymbol{x})$ that maps $\Sigma_{(\boldsymbol{f},\mathbf{g})}$ into a linear system

$$\Sigma_{(\mathbf{A},\mathbf{B})} : \dot{\boldsymbol{z}} = \mathbf{A}\boldsymbol{z} + \mathbf{B}u = \mathbf{A}\boldsymbol{z} + \mathbf{B}_1 u_1 + \cdots + \mathbf{B}_n u_m?$$

Indeed, he showed that the problem is solvable (with a controllable pair $(\mathbf{A}, \mathbf{B})$) if and only if the set $\mathcal{L}(\boldsymbol{f}, \mathbf{g}) = \operatorname{span}\big\{\mathbf{g}_i, ad_{\boldsymbol{f}}(\mathbf{g}_i), \ldots, ad_{\boldsymbol{f}}^{n-1}(\mathbf{g}_i),\ i = 1, \ldots, m\big\}$ associated with $\Sigma_{(\boldsymbol{f},\mathbf{g})}$ is full rank n and pairwise commutative under the Lie bracket $[X, Y] = (\partial Y/\partial \boldsymbol{x})X - (\partial X/\partial \boldsymbol{x})Y$.

The finding of the linearizing diffeomorphism is then reduced in solving a system of partial differential equations

$$\begin{cases} \dfrac{\partial \varphi}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) = \dfrac{\partial \varphi}{\partial \boldsymbol{x}_1}\boldsymbol{f}_1(\boldsymbol{x}) + \cdots + \dfrac{\partial \varphi}{\partial \boldsymbol{x}_n}\boldsymbol{f}_n(\boldsymbol{x}) = A\varphi(\boldsymbol{x}) \\ \dfrac{\partial \varphi}{\partial \boldsymbol{x}}\mathbf{g}_1(\boldsymbol{x}) = \dfrac{\partial \varphi}{\partial \boldsymbol{x}_1}\mathbf{g}_{\boldsymbol{1}1}(\boldsymbol{x}) + \cdots + \dfrac{\partial \varphi}{\partial \boldsymbol{x}_n}\mathbf{g}_{\boldsymbol{1}n}(\boldsymbol{x}) = \mathbf{B}_1 \\ \cdots \\ \dfrac{\partial \varphi}{\partial \boldsymbol{x}}\mathbf{g}_m(\boldsymbol{x}) = \dfrac{\partial \varphi}{\partial \boldsymbol{x}_1}\mathbf{g}_{\boldsymbol{m}1}(\boldsymbol{x}) + \cdots + \dfrac{\partial \varphi}{\partial \boldsymbol{x}_n}\mathbf{g}_{\boldsymbol{m}n}(\boldsymbol{x}) = \mathbf{B}_m. \end{cases}$$

For single-input systems ($m = 1$), some algorithms have already been proposed in the literature. In Mullhaupt (2006) and Willson, Mullhaupt, and Bonvin (2009) a method based on successive integrations of differential one forms has been given. It relies on successive rectification of one vector field at a time via the characteristic method using quotient manifolds in order to reduce, at each step, the dimension of the system by one. More recently we proposed an algorithm for state linearization in Tall (2009a) and another one for feedback linearization in Tall (2009b) via Lie series of vector fields (see Tall, 2010b). Those algorithms give rise to a sequence of affine k-linear (resp. k-feedback) systems whose last $(n-k)$ components are linear (resp. in feedback form). This paper generalizes the results of Tall (2010b) to multi-input control systems and is a journal version of the conference paper (Tall, 2010c). We extend the explicit solving of the Flow-Box Theorem to a particular case of Frobenius Theorem, that is, for a commutative, full rank set of vector fields, we provide an algorithm allowing to find change of coordinates that simultaneously rectify the whole set of vector fields (the vector fields associated with the new system are constant). Though explicit formulas in terms of power series of functions are provided, the characteristic method (Isidori, 1995; Mullhaupt, 2006; Willson et al., 2009) or any direct method that rectifies a vector field or set of vector fields can be applied. The importance of the algorithm being the fact that it lays out the steps and the vector fields to be rectified, as well as necessary and sufficient conditions for this to be done, at each step. Let it be mentioned that Gardner and Shadwick (1992) proposed an algorithm, called GS algorithm, that is based on integrating Pfaffian systems. For the class of feedforward and strict feedforward systems, other algorithms involving differentiation and integration of functions were provided in Krstic (2004) and Tall (2010a,d). The organization of the paper is as follows. We first give notations and definitions. Section 2 deals with the main result of state linearization immediately followed by a constructive algorithm. The algorithm is based on an iterative application of a particular case of the Frobenius Theorem whose constructive proof is given in the Appendix. Section 3 deals with examples illustrating the algorithm.

Notations and definitions

We consider *analytic* multi-input control systems

$$\Sigma_{(\boldsymbol{f},\mathbf{g})} : \dot{\boldsymbol{x}} : \boldsymbol{f}(\boldsymbol{x}) + \mathbf{g}(\boldsymbol{x})u = \boldsymbol{f}(\boldsymbol{x}) + \mathbf{g}_1(\boldsymbol{x})u_1 + \cdots + \mathbf{g}_m(\boldsymbol{x})u_m,$$

where $\boldsymbol{f} := (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n)^\top$ and $\mathbf{g}_i := (\mathbf{g}_{\boldsymbol{i}1}, \ldots, \mathbf{g}_{\boldsymbol{i}n})^\top$ are analytic vector fields on $\mathbb{R}^n$, $\boldsymbol{f}(0) = 0$ and the control vector fields satisfy $\operatorname{rank}\{\mathbf{g}_1(\boldsymbol{x}), \ldots, \mathbf{g}_m(\boldsymbol{x})\} = m$.

The system $\Sigma_{(\boldsymbol{f},\mathbf{g})}$ is called linearly controllable if

$$\Sigma_{(\mathbf{A},\mathbf{B})} : \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}u = \mathbf{A}\boldsymbol{x} + \mathbf{B}_1 u_1 + \cdots + \mathbf{B}_m u_m$$

where $\mathbf{A} = \frac{\partial f}{\partial \boldsymbol{x}}(0)$, $\mathbf{B} = \mathbf{g}(0)$ is controllable, i.e., there exist positive integers (Brunovský controllability indices) $r_1 \geq 1, \ldots, r_m \geq 1$ with $r_1 + \cdots + r_m = n$ such that

$$\dim \operatorname{span}\left\{\mathbf{A}^j \mathbf{B}_i,\ 0 \leq j \leq r_i - 1,\ 1 \leq i \leq m\right\} = n$$

and put $\boldsymbol{x} = (\boldsymbol{x}_1^\top, \ldots, \boldsymbol{x}_m^\top)^\top$ with $\boldsymbol{x}_i^\top = (\boldsymbol{x}_{\boldsymbol{i}1}, \ldots, \boldsymbol{x}_{\boldsymbol{i}r_i})$.

For a complete description and geometric interpretation of the Brunovský controllability indices we refer to the literature (Antsaklis & Michel, 1997; Hunt & Su, 1981; Isidori, 1995; Jakubczyk & Respondek, 1980; Kailath, 1980; Nijmeijer & van der Schaft, 1990). Without loss of generality we assume that $\mathbf{A} = \operatorname{diag}\{\mathbf{A}_1, \ldots, \mathbf{A}_m\}$ and $\mathbf{B} = (\mathbf{B}_1 \cdots \mathbf{B}_m) = \operatorname{diag}\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ with each pair $(\mathbf{A}_i, \mathbf{b}_i)$ in Brunovský form of dimension r_i and for simplicity that $r_1 = \cdots = r_m = r$. The case of distinct indices follows easily by extending the system appropriately. Let $0 \leq k \leq r$ be an integer.

Definition 1. $\Sigma_{(\boldsymbol{f},\mathbf{g})}$ is called *quasi* k-linear and we put

$$\Sigma_{(\boldsymbol{f},\mathbf{g})} \triangleq \Sigma_{(\boldsymbol{f}^k,\mathbf{g}^k)} : \dot{\boldsymbol{x}} = \boldsymbol{f}^k(\boldsymbol{x}) + \mathbf{g}_1^k(\boldsymbol{x})u_1 + \cdots + \mathbf{g}_m^k(\boldsymbol{x})u_m,$$

if for any $1 \leq i \leq m$, we have $\mathbf{g}_i^k(\boldsymbol{x}) = \mathbf{B}_i$, and

$$\boldsymbol{f}^k(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} + \mathbf{F}^k(\boldsymbol{x}_{\boldsymbol{1}1}, \ldots, \boldsymbol{x}_{\boldsymbol{1}k+1}, \ldots, \boldsymbol{x}_{\boldsymbol{m}1}, \ldots, \boldsymbol{x}_{\boldsymbol{m}k+1})$$

with $\mathbf{F}^k(0) = 0$ and $\frac{\partial \mathbf{F}^k}{\partial \boldsymbol{x}}(0) = 0$.

A quasi k-linear system appears hence to be a linear system perturbed by terms that depend only on the variables $\boldsymbol{x}_{\boldsymbol{i},1}, \ldots, \boldsymbol{x}_{\boldsymbol{i},k+1}$ for $1 \leq \boldsymbol{i} \leq m$. The importance of quasi k-linear systems is demonstrated next.

2. Main results

The main result of this paper is an algorithm transforming a system into a sequence of quasi k-linear systems. All diffeomorphisms are local around the equilibrium point.

Theorem 2.1. *A linearly controllable system*

$$\Sigma_{(\boldsymbol{f}^r,\mathbf{g}^r)} : \dot{\boldsymbol{x}} = \boldsymbol{f}^r(\boldsymbol{x}) + \mathbf{g}_1^r(\boldsymbol{x})u_1 + \cdots + \mathbf{g}_m^r(\boldsymbol{x})u_m, \quad \boldsymbol{x} \in \mathbb{R}^n$$

is $\mathcal{S}$-linearizable if and only if there exist a sequence $\varphi^r, \ldots, \varphi^1$ of explicit diffeomorphisms that gives rise to a sequence $\Sigma_{(\boldsymbol{f}^{r-1},\mathbf{g}^{r-1})}, \ldots, \Sigma_{(\boldsymbol{f}^0,\mathbf{g}^0)}$ of quasi k-linear systems s.t. $\Sigma_{(\boldsymbol{f}^{k-1},\mathbf{g}^{k-1})} = {\varphi^k}_* \Sigma_{(\boldsymbol{f}^k,\mathbf{g}^k)}$.

The quasi k-linear $\Sigma_{(\boldsymbol{f}^k,\mathbf{g}^k)}$ can be mapped into a quasi $(k-1)$-linear $\Sigma_{(\boldsymbol{f}^{k-1},\mathbf{g}^{k-1})}$ if and only if

$$(\mathcal{SL}_k) \triangleq \begin{cases} \text{(a)}\ \dfrac{\partial^2 \boldsymbol{f}^k(\boldsymbol{x})}{\partial \boldsymbol{x}_{\boldsymbol{i}k+1}\partial \boldsymbol{x}_{\boldsymbol{j}k+1}} = 0, & 1 \leq \boldsymbol{i}, \boldsymbol{j} \leq m \\ \text{(b)}\ \left[\dfrac{\partial \boldsymbol{f}^k}{\partial \boldsymbol{x}_{\boldsymbol{i}k+1}}, \dfrac{\partial \boldsymbol{f}^k}{\partial \boldsymbol{x}_{\boldsymbol{j}k+1}}\right] = 0, & 1 \leq \boldsymbol{i}, \boldsymbol{j} \leq m. \end{cases}$$

Moreover, in the coordinates $\boldsymbol{z} = \varphi^1 \circ \cdots \circ \varphi^r(\boldsymbol{x})$:

$$\boldsymbol{z} = (\boldsymbol{z}_{\boldsymbol{1}1}, \ldots, \boldsymbol{z}_{\boldsymbol{1}r}, \boldsymbol{z}_{\boldsymbol{2}1}, \ldots, \boldsymbol{z}_{\boldsymbol{2}r}, \ldots, \boldsymbol{z}_{\boldsymbol{m}1}, \ldots, \boldsymbol{z}_{\boldsymbol{m}r})$$

the system $\Sigma_{(\boldsymbol{f}^0,\mathbf{g}^0)}$ takes the linear form

$$\Sigma_{(\boldsymbol{f}^0,\mathbf{g}^0)} \triangleq \Sigma_{(\mathbf{A},\mathbf{B})} : \dot{\boldsymbol{z}} = \mathbf{A}\boldsymbol{z} + \mathbf{B}u = \mathbf{A}\boldsymbol{z} + \mathbf{B}_1 u_1 + \cdots + \mathbf{B}_m u_m.$$