Automatica 50 (2014) 747–755

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

A regime-switching model with the volatility smile for two-asset European options☆

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ARTICLE INFO

Article history:
Received 15 November 2011
Received in revised form
15 November 2013
Accepted 27 November 2013
Available online 1 February 2014

Keywords:
Regime-switching model
Finite difference method
Operator splitting method
Volatility smile

ABSTRACT

In this paper, we consider a numerical European-style option pricing method under two regime-switching underlying assets depending on the market regime. For a risk neutral market condition, we consider regime-switching model with two assets using a Feynman–Kac type formula. And to solve the option problem with regime-switching model, we apply an operator splitting method. Numerical examples show the volatility smile and the volatility term structure under varying parameters on a two state regime switching model.

1. Introduction

The Black–Scholes (BS) formula for option pricing is widely applied to the pricing of numerous European options; see Haug (1997). The underlying securities of the Black–Scholes formula are supposed to be geometric Brownian motions that contain pairs of two parameters, the expected rate of return and the volatility. Both parameters are assumed to be constants in the general Black–Scholes model, and these assumptions are not applicable to option pricing in real markets. To overcome the shortfall of the BS model, the volatility smile and term structure are used to capture the change in volatility in terms of the price and the maturity of a security.

The regime-switching model is an alternative model to illustrate the stochastic volatility. Since stock parameters practically are depended on the market mode that switches among a finite number of states, we naturally allow the key parameters of the underlying assets to reflect a random market environment.

The regime-switching model is invoked to formulate such parameters that are governed by the random market mode. In 1989, the regime-switching model was first introduced by Hamilton (1989) to describe a regime-switching time series. In option pricing, regime-switching model has been applied in various other problems. Zhang (2001) used this model to calculate an optimal selling rule and Yin and Zhang (1998) applied this in portfolio management. Also, Yin and Zhou (2003) studied a dynamic Markowitz problem for a market consisting of one bank account and multiple stocks.

In this study, we consider an efficient and accurate numerical method of a regime-switching model for European options (Kim, Jang, & Lee, 2008). Among several numerical methods for pricing of options with multi-underlying assets, the operator splitting (OS) scheme will be used: see Duffy (2006) and Ikonen and Toivanen (2004). In general, standard finite difference methods (FDM) do not work well for discrete options due to non-smooth payoffs or discontinuous derivatives at the exercise price. On the other hand, the OS scheme does not result in problematic oscillations due to the source term (Jeong & Kim, 2013). The main purpose of this paper is to observe the volatility smile and term structure of a regime-switching model by using an efficient and accurate numerical method. This work is an extension of the earlier one-dimensional study of Buffington and Elliott (2002).

This paper is organized as follows. In Section 2, we briefly introduce the risk-neutral valuation method and regime-switching.

☆ The first author (J.S. Kim) was supported by the National Institute for Mathematical Sciences (NIMS) grant funded by the Korea government (No. A21301). The corresponding author (D.H. Shin) was supported by Inha University and the Institute of Basic Science Korea University. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Editor Berç Rüstem.

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0005-1098/$ – see front matter
http://dx.doi.org/10.1016/j.automatica.2013.12.019

In Section 3, we discuss the Feynman–Kac type formula that is satisfied by the option valuation function. We describe the algorithm of the OS method for the formula at the end of this section. In Section 4, we perform convergence test and comparison study of ADI and OS methods. The volatility smile and term structure with a simple regime-switching model are reported in Section 5. In this section, we propose an algorithm for finding the implied volatility and by using this algorithm, we carry out several numerical parameter tests. We conclude this study in Section 6.

2. Risk neutral pricing

Standard research in derivative pricing follows the idea that the expected rate of return of all securities has the same risk-free interest rate in an appropriate probability space. We call the probability space the risk-neutral world, and the discount asset price is a martingale in this world.

Let $(\Omega, \mathcal{F}, P)$ denote the probability space and $\{\alpha(t)\}$ denote a continuous-time Markov chain with state space $\mathcal{M} = \{1, 2, \ldots, m\}$. In a regime-switching model, $\{\alpha(t)\}$ represents the market regime that determines the rate of return and volatility. Then, for example, the price of a stock $X(t)$ at time t is governed by:

$$dX(t) = X(t)\left[\mu(\alpha(t))dt + \sigma(\alpha(t))dw(t)\right], \quad \text{for } 0 \le t \le T, \qquad X(0) = X_0.$$

Let $Q = (q_{ij})_{m\times m}$ be the generator of $\alpha(t)$ with $q_{ij} \ge 0$ for $i \ne j$ and $\sum_{j\ne i}^m q_{ij} = -q_{ii}$ for each $i \in \mathcal{M}$. For any function f on $\mathcal{M}$, we denote $Qf(\cdot)(i) := \sum_{j=1}^m q_{ij} f(j)$.

In this paper, one of our objectives is to price European style options under regime-switching multi-underlying assets. Consider $X_k(t)$ as the price of stock k at time t with

$$dX_k(t) = X_k(t)\left[\mu_k(\alpha(t))dt + \sigma_k(\alpha(t))dw_k(t)\right], \quad \text{for } 0 \le t \le T,\ k = 1, 2, \ldots, d, \quad \text{and} \quad X_k(0) = X_{k0}, \tag{1}$$

where $\mu_k(i)$ and $\sigma_k(i)$ respectively represent the expected rate of return for X_k and the volatility of the stock price X_k at regime $i \in \mathcal{M}$, and $w_k(\cdot)$ denotes the standard Brownian motion. The Wiener processes are correlated by

$$\langle dw_k, dw_l\rangle = \rho_{kl}dt, \quad \text{for } \rho_{kl} \in [-1, 1].$$

In order to introduce derivative pricing in the risk neutral market, we also discuss the martingale measure characterized in Lemma 1. Assume that $X_0, \alpha(\cdot)$, and $w_k(\cdot)$ are mutually independent, and $\sigma_k^2(i) > 0$ for all $i \in M$. Let $\mathcal{F}_t$ denote the sigma field generated by $\{(\alpha(s), w_k(s)) : 0 \le s \le t\}$, and let $r > 0$ denote the risk-free rate. For $0 \le t \le T$, let

$$Z_t := \exp\left[\int_0^t \beta_k(s)dw_k(s) - \frac{1}{2}\int_0^t \beta_k^2(s)ds\right],$$

where

$$\beta_k(s) := \frac{r - \mu_k(\alpha(s))}{\sigma_k(\alpha(s))}.$$

Then, in lieu of Ito's rule,

$$\frac{dZ_t}{Z_t} = \beta_k(t)dw_k(t)$$

and Z_t is a local martingale with

$$E[Z_t] = 1, \quad 0 \le t \le T.$$

We define an equivalent measure $\widetilde{P}$ with the following

$$\frac{d\widetilde{P}}{dP} = Z_T.$$

Therefore Lemma 1 is a generalized Girsanov's theorem for Markov-modulated processes.

Lemma 1. *(1) Let $\widetilde{w}_k(t) := w_k(t) - \int_0^t \beta_k(s)\,ds (k = 1 : d)$. Then, $\widetilde{w}_k(t)$ is a $\widetilde{P}$-Brownian motion.*

(2) X_0, $\alpha(\cdot)$, and $\widetilde{w}_k(\cdot)$ are mutually independent under $\widetilde{P}$.

(3) Let $\mathbf{X}(t) := (X_1(t), X_2(t), \ldots, X_d(t))$, $c \le t$, and $\sigma_{X_k}(i) :=$ the volatility of stock X_k at regime i. Dynkin's formula holds: for any smooth function $\mathcal{F}(t, \mathbf{X}, i)$, we have

$$\mathcal{F}(t, \mathbf{X}(t), \alpha(t)) = \mathcal{F}(c, \mathbf{X}(c), \alpha(c)) + \int_c^t \mathcal{A}\mathcal{F}(s, \mathbf{X}(s), \alpha(s))ds + M(t) - M(c),$$

where $M(\cdot)$ is a $\widetilde{P}$-martingale and $\mathcal{A}$ is a generator given by

$$\mathcal{A}\mathcal{F} = \frac{\partial}{\partial t}\mathcal{F}(t, \mathbf{X}, i) + \sum_{k=1}^d rX_k\frac{\partial}{\partial X_k}\mathcal{F}(t, \mathbf{X}, i) + \frac{1}{2}\sum_{k=1}^d\sum_{l=1}^d \rho_{kl}(i)\sigma_{X_k}(i)\sigma_{X_l}(i)X_kX_l\frac{\partial^2}{\partial X_k\partial X_l}\mathcal{F}(t, \mathbf{X}, i) + Q\mathcal{F}(t, \mathbf{X}, \cdot)(i),$$

where $\rho_{kk} = 1$ for $1 \le k \le d$.

Proof. See Chapter 14 in Yao, Zhang, and Zhou (2006). □

From Lemma 1 and this point of view of Fouque, Papanicolaou, and Sircar (2000) and Hull (2000), $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \widetilde{P})$ defines a risk-neutral world. And $e^{-rt}X(t)$ is a $\widetilde{P}$-martingale.

3. A numerical approach with OS methods

In this paper, we consider European style option pricing under two regime-switching underlying assets $X_1(t)$ and $X_2(t)$. Let $x := X_1(t)$, $y := X_2(t)$, and $\mathbf{U}(x, y, t, i)$ be the values of a European style call option with two underlying assets with regime i for $i = 1, 2$. Using a Feynman–Kac formula, a partial difference equation with respect to $\mathbf{U}(x, y, t) = (u(x, y, t), v(x, y, t))^T$ is derived as follows:

$$\frac{\partial \mathbf{U}}{\partial t} + rx\frac{\partial \mathbf{U}}{\partial x} + ry\frac{\partial \mathbf{U}}{\partial y} - r\mathbf{U} + \frac{1}{2}(\sigma_x x)^2\frac{\partial^2 \mathbf{U}}{\partial x^2} + \frac{1}{2}(\sigma_y y)^2\frac{\partial^2 \mathbf{U}}{\partial y^2} + \rho_{xy}\sigma_x\sigma_y xy\frac{\partial^2 \mathbf{U}}{\partial x\partial y} + Q\mathbf{U} = 0,$$

where $Q = \begin{pmatrix} -\lambda^u & \lambda^u \\ \lambda^v & -\lambda^v \end{pmatrix}$ and λ^u, λ^v represent jumping rates for u and v, respectively.

Then, by each component of $\mathbf{U}$, we have the following system:

$$\frac{\partial u}{\partial t} + r^u x\frac{\partial u}{\partial x} + r^u y\frac{\partial u}{\partial y} - r^u u + \frac{1}{2}(\sigma_x^u x)^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(\sigma_y^u y)^2\frac{\partial^2 u}{\partial y^2} + \rho_{xy}^u\sigma_x^u\sigma_y^u xy\frac{\partial^2 u}{\partial x\partial y} + \lambda^u(v - u) = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + r^v x\frac{\partial v}{\partial x} + r^v y\frac{\partial v}{\partial y} - r^v v + \frac{1}{2}(\sigma_x^v x)^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}(\sigma_y^v y)^2\frac{\partial^2 v}{\partial y^2} + \rho_{xy}^v\sigma_x^v\sigma_y^v xy\frac{\partial^2 v}{\partial x\partial y} + \lambda^v(u - v) = 0. \tag{3}$$

The terminal conditions $u(x, y, T) = v(x, y, T)$ are given by $\Lambda(x, y)$.