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The solution of linear mechanical systems in terms of path superposition

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ABSTRACT

We prove that the solution of any linear mechanical system can be expressed as a linear combination of signal transmission paths. This is done in the framework of the Global Transfer Direct Transfer (GTDT) formulation for vibroacoustic problems. Transmission paths are expressed as powers of the transfer matrix. The key idea of the proof is to generalise the Neumann series of the transfer matrix – which is convergent only if its spectral radius is smaller than one – into a modified Neumann series that is convergent regardless of the eigenvalues of the transfer matrix. The modification consists in choosing the appropriate combination coefficients for the powers of the transfer matrix in the series. A recursive formula for the computation of these factors is derived. The theoretical results are illustrated by means of numerical examples. Finally, we show that the generalised Neumann series can be understood as an acceleration (i.e. convergence speedup) of the Jacobi iterative method.

1. Introduction

Vibroacoustic problems are very often not easy to visualise and understand. Moreover, the availability of experimental measurements is limited by operational costs and times. For these reasons the intuition of engineers/physicists/acousticians has always played an important role during the design process. A key concept is the *transmission path* of sound and vibrations. This has remained an intuitive idea rather than a properly defined and well established concept.

The first attempts to quantify the contribution of subsystems, even if they never spoke about paths, can be found in [1] and later works [2–6]. They were motivated by the need in the automotive industry to characterise how the noise generated by the engine or in the moving parts of the vehicle were transmitted to the cabin. This method is nowadays known as Operational Transfer Path Analysis (OTPA), [7,8].

Paths are implicitly defined and quantified in [9]. The role of paths in more specific situations was analysed in [10,11], where the interest is focused in the characterisation of the connectivity between system parts. This method is known as Global Transfer Direct Transfer (GTDT) method in the papers or as Advanced Transfer Path Analysis (ATPA) in the industry, where it has been widely used in many applications, such as railways [12]. Some academic tests can be also found in [13,14].

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List of symbols

$\mathbf{A}$	system matrix
α_i, $\beta_{j,i}$, γ_i	parameters in generalised Neumann series
$\mathbf{b}$	vector of forces or excitations
$\mathbf{D}$	diagonal part of matrix $\mathbf{A}$
$\mathbf{I}$	identity matrix
$\mathbf{L}$	strict lower triangular part of matrix $\mathbf{A}$
λ	eigenvalues (typically of matrix $\mathbf{T}$)
n	problem dimension
$\mathbf{S}_{j,m}$	partial sum of order j (j modified parameters) and $m+1$ terms
$\mathbf{T}$	transfer matrix
$\mathbf{T}^G$	global transfer matrix
$\mathbf{U}$	strict upper triangular part of matrix $\mathbf{A}$
$\mathbf{x}$	vector of unknowns or signals

A comprehensive classification of the methods and historical overview can be found in [15]. Paths have also been defined and quantified in a Statistical Energy Analysis (SEA) framework [16–18].

Other applications of the path concept can be found in the literature. An analysis of the paths that contribute more to the system response by means of graph theory is presented in [19,20]. Path analysis was simplified considering forward paths only in [18]. A comparison of path analysis with other methods was reported in [21].

In spite of the clear applicability of path analysis to practical situations, a theoretical question remains open: a proof of completeness. In other words, the possibility of fully describing the solution of a mechanical transfer problem by means of the superposition of transmission paths. This theoretical question is addressed here, by using concepts and tools of numerical linear algebra (see textbooks [22,23] for background material).

It was clearly demonstrated in [24,18] that a solution of a mechanical problem can be described by means of the Neumann series[1] of the transfer matrix $\mathbf{T}$ (the powers of $\mathbf{T}$ are a representation of paths of different order in the mechanical system). The series has strict convergence conditions, which in practise mean that the solution description through transmission paths and Neumann series[2] is not always possible.

The issue of the completeness of the solution description has also been addressed in [19,25–27]. Ref. [25] relates the convergence of the series with the damping of the systems because the energy of undamped systems permanently excited would grow indefinitely. However, this does not explain the situations when, even with damping, the series diverge. For this reason [19] claims that other conditions to ensure the convergence are required in addition to the existence of damping. The drawback of the divergence of the solution expressed as a series also appears if the problem is not strictly formulated in terms of the transfer matrix and its powers. This can be seen in [26] for the matrix of coupling loss factors and in [27] for the coupling eigenvalues.

The goals and achievements of this research are as follows:

- To provide a definition of what a path is.
- To prove the possibility of expressing the solution of all linear problems in terms of paths (especially applied to vibroacoustics). The proof is done in the framework of the Direct and Global Transfer Matrix formulation of the problem. The final result is a generalisation of the Neumann series.
- To derive a practical recursive methodology that allows the computation of the solution based on the transfer matrix of the problem. The goal is also to provide a closed-form expression of the solution as simple as possible.
- To illustrate this methodology with numerical examples.
- To explore the relationship between the proposed approach and Jacobi iterative method for linear systems. The generalised Neumann series can be understood as an acceleration (i.e. convergence speedup) of the Jacobi method.

The remainder of the paper is organised as follows. Some key concepts such as the notion of path and the transfer matrix are defined in Section 2. A precise and explicit definition of path is given. The theoretical core of the research is presented in Section 3. It includes the proof of the existence of an expression of the system solution based on a linear combination of paths. A general methodology to compute the combination factors in the generalised Neumann series is provided. Numerical examples that illustrate the theoretical results are shown in Section 4. The concluding remarks of Section 5 close the paper.

2. Definitions

2.1. Physical considerations on the 'path' concept

Various methods based on path analysis are useful to find engineering solutions in vibroacoustic problems [15]. Nevertheless the 'path' concept is often not defined in a rigorous way.

[1] In honour of Karl Gottfried Neumann (1832–1925).

[2] A Neumann series has the form $\sum_{k=0}^{\infty}\mathbf{T}^k$.