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An extended harmonic balance method based on incremental nonlinear control parameters

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A R T I C L E I N F O

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A B S T R A C T

A new formulation for calculating the steady-state responses of multiple-degree-of-freedom (MDOF) non-linear dynamic systems due to harmonic excitation is developed. This is aimed at solving multi-dimensional nonlinear systems using linear equations. Nonlinearity is parameterised by a set of 'non-linear control parameters' such that the dynamic system is effectively linear for zero values of these parameters and nonlinearity increases with increasing values of these parameters. Two sets of linear equations which are formed from a first-order truncated Taylor series expansion are developed. The first set of linear equations provides the summation of sensitivities of linear system responses with respect to non-linear control parameters and the second set are recursive equations that use the previous responses to update the sensitivities. The obtained sensitivities of steady-state responses are then used to calculate the steady state responses of non-linear dynamic systems in an iterative process. The application and verification of the method are illustrated using a non-linear Micro-Electro-Mechanical System (MEMS) subject to a base harmonic excitation. The non-linear control parameters in these examples are the DC voltages that are applied to the electrodes of the MEMS devices.

1. Introduction

Vibration analysis of structures containing nonlinearities is one of the important topics in structural engineering problems. There are many practical engineering components that are modelled as nonlinear oscillatory systems. In most cases, the nonlinear dynamics of these systems have been investigated through numerical methods such as the Newmark method, the shooting method, the differential quadrature method and the Adomian decomposition method [1–4]. Using these simulations to study the effect of different parameters on the dynamics of the system is always time consuming, particularly for multi-dimensional non-linear systems and the cases where the sensitivities of responses are required. Therefore obtaining the steady state solution of multiple-degree-of-freedom non-linear dynamic systems is of great importance in this field. A comprehensive account of nonlinear structural dynamics and control is given by Wagg and Neild [5].

In order to investigate the analytical solution of nonlinear structures, different techniques have been applied in the literature. The Max-min [6], the parameter-expanding approach [7], frequency-amplitude formulation [8], the Variational Iteration [9], perturbation techniques [10–12], the iteration perturbation [13], the Homotopy Analysis [14,15], the Energy Balance analysis

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[16], the harmonic balance [17], the equivalent linearization method (ELM) [18,19] and the Extended Lindstedt-Poincare approach [20] are some examples of these techniques. Each of these methods has some strong points and some weakness. The perturbation methods have been used for both weakly and strongly nonlinear problems (e.g. [21,22]) and they are expressed by a series of perturbation quantities. Based on these quantities, the original nonlinear equations are replaced by linear equations (sometimes even nonlinear), which are specified by the original equation and also by the place where the perturbation quantities appear. Methods such as Homotopy Analysis, Variational Iteration, Energy Balance and harmonic balance are also suitable for dealing with strong nonlinear problems. Qian et al. [14] studied the oscillation of a MEMS microbeam with strong nonlinearity by means of the Homotopy Analysis Method. They demonstrated that the method has good performance in investigating the nonlinear equation of the model studied in the paper. Fesanghary et al. [23] utilised a variational iterative method and proposed a new analytical approximation for the Duffing-harmonic oscillator. Their solution was valid in the whole range of oscillation amplitude variations; but it contained many harmonic terms. Younesian et al. [8] investigated the generalized Duffing equation using a frequency-amplitude formulation and energy balance method. They obtained the general solution for any arbitrary type of nonlinearity and showed that these two techniques are quite reliable even in strongly nonlinear systems. Peng et al. [24] applied the harmonic balance method to study the effects of cubic nonlinear damping on the performance of passive vibration isolators. Harmonic balance has been used to predict the steady-state solution of structures with different types of nonlinearity and also can be used for identification and health monitoring of nonlinear systems. There are several developments of the harmonic balance method such as incremental harmonic balance [25,26], Newton harmonic balance [27], adaptive harmonic balance [28], residue harmonic balance [29], and Global residue harmonic balance [30]. Although these methods have been successfully used to obtain analytical solutions of different non-linear problems, there are few applications to multi-degree of freedom non-linear dynamic problems. In these problems, applying the aforementioned methods requires the solution of complicated non-linear algebraic equations which is really time consuming. Furthermore, in order to find the sensitivity of responses to non-linear parameters, additional computations are required.

This paper proposes an extended harmonic balance method for the steady-state solution of non-linear multiple-degree-of-freedom dynamic problems based on incremental nonlinear control parameters. The method only requires the solution of linear equations for the nonlinear problem. It also provides the sensitivities of the solution with respect to the nonlinear control parameters. The nonlinear control parameters are those with which the non-linearity in the model is triggered. This property of nonlinear control parameters can be implemented in the solution of a nonlinear problem. They are incremented from zero to one (note that the parameters are normalised so that the maximum values are unity) and a linear equation giving the sensitivities of the responses with respect to the parameters is obtained at each increment. Using these sensitivities, the solution at each step can be calculated through the solution at the previous increment. The method starts from the linear system and continues until all nonlinear control parameters reach unity. The major advantages of the proposed method include the capability of solving smooth multi-dimensional nonlinear dynamic systems using linear equation solvers and the ability to obtain sensitivities that are useful in inverse problems such as vibration control and robust design. The application and verification of the method is demonstrated in a non-linear Micro-Electro-Mechanical System (MEMS) subject to a base harmonic excitation. The non-linearity is due to the electrostatic forces and the nonlinear control parameters are the DC voltages. The method is verified using numerical integration and some interesting results from the frequency responses of sensitivities are demonstrated and discussed.

2. Theory

Consider a damped structural dynamic system, defined on the domain $\mathcal{D} \in \mathbb{R}^d \left(d \le 3\right)$ with piecewise Lipschitz boundary $\partial\mathcal{D}$, subject to an externally applied harmonic excitation with the distributed forcing function $f\left(\mathbf{r}\right)$. The governing equation may be cast in the form of the following general partial differential equation

$$\rho\left(\mathbf{r}\right)\frac{\partial^2 w\left(\mathbf{r},t\right)}{\partial t^2} + c\left(\mathbf{r}\right)\frac{\partial w\left(\mathbf{r},t\right)}{\partial t} + k\left(\mathbf{r}\right)\Gamma\left(w\left(\mathbf{r},t\right)\right) + f_{NL}\left(w\left(\mathbf{r},t\right),\bar{\boldsymbol{\theta}}\right) = \frac{f\left(\mathbf{r}\right)}{2}\exp\left(i\Omega t\right) + cc. \tag{1}$$

with the usual mixture of Cauchy and Neumann boundary conditions on $\partial\mathcal{D}$. In the above equation (Eq. (1)), $w\left(\mathbf{r},t\right)$ is the displacement variable, where $\mathbf{r} \in \mathcal{D}$ is the spatial position vector and $t \in [0, T]$ is time, $\rho\left(\mathbf{r}\right)$ is the distributed mass density, $c\left(\mathbf{r}\right)$ is the distributed damping coefficient, $k\left(\mathbf{r}\right)$ is the distributed stiffness parameter, Γ is a general differential operator, $f_{NL}\left(w\left(\mathbf{r},t\right),\bar{\boldsymbol{\theta}}\right)$ is the non-linear restoring force function with $\bar{\boldsymbol{\theta}} \in \mathbb{R}^m$ ($\bar{\boldsymbol{\theta}}$ includes the normalised non-linear control parameters $\mathbf{0} \le \bar{\boldsymbol{\theta}} \le \mathbf{1}$ and $f_{NL}\left(w,\mathbf{0}\right) = 0$), $i = \sqrt{-1}$, Ω is the excitation frequency and cc. denotes complex conjugate. Hereafter $w\left(\mathbf{r},t\right)$, $f\left(\mathbf{r}\right)$, $\rho\left(\mathbf{r}\right)$, $c\left(\mathbf{r}\right)$ and $k\left(\mathbf{r}\right)$ are replaced by w, f, ρ, c and k for reasons of simplicity.

In this paper, it is assumed that the vector of nonlinear restoring force can be reasonably approximated by a limited number of terms of its Taylor series, i.e. H terms,

$$f_{NL}\left(w,\bar{\boldsymbol{\theta}}\right) \approx \sum_{h=0}^{H} \alpha_h\left(\bar{\boldsymbol{\theta}}\right) w^h \tag{2}$$