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Brief paper

A distributed algorithm for average consensus on strongly connected weighted digraphs☆

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ABSTRACT

In this work we propose a distributed algorithm to solve the discrete-time average consensus problem on strongly connected weighted digraphs (SCWDs). The key idea is to couple the computation of the average with the estimation of the left eigenvector associated with the zero eigenvalue of the Laplacian matrix according to the protocol described in Qu et al. (2012). The major contribution is the removal of the requirement of the knowledge of the out-neighborhood of an agent, thus paving the way for a simple implementation based on a pure broadcast-based communication scheme.

1. Introduction

In the past decades, multi-agent systems have gained an increasing interest from the control theory community. Applications range from transportation to environmental monitoring (see Oh, Schenato, Chen, & Sastry, 2007). Distributed algorithms to estimate the status of the system are essential in this context, as they can help the agents modify their behavior in order to improve the global response (Gasparri, Fiorini, Di Rocco, & Panzieri, 2012; Ren & Beard, 2007).

Within several of the works related to this topic, the communication among agents is modeled using an undirected communication graph (see Mesbahi & Egerstedt, 2010 and the references therein). This is founded on the assumptions that the communication is isotropic, i.e., the employed antenna radiates its power uniformly in all directions and that its range is the same for all the agents in the network. Therefore, if an agent can communicate with another one, the opposite is possible as well. However, this assumption is not always realistic in a real world scenario due, for example, to environmental effects or the radiation pattern of the agents (Luthy, Grant, & Henderson, 2007).

In this work, we consider a more general scenario where the communication among the agents is modeled as a directed graph. In particular, two different communication schemes can be considered, that is point-to-point or broadcast. We refer to point-to-point as a communication mechanism where an agent (sender) transmits a specific message to another agent (receiver), picking out exactly that agent among all of his neighbors. Note that this communication scheme requires the sender to know the neighbors it is going to send the messages to, i.e., each agent must know its out-neighborhood. In contrast, we refer to broadcast as a communication mechanism where an agent (sender) can simply transmit a message which will be received by any other agent (receivers) within its range of transmission. In our opinion, this latter communication mechanism represents a better choice since it can be more easily implemented and provides a higher robustness to the system.

Our contribution is a novel distributed algorithm to compute the average consensus over any strongly connected weighted digraph, which can be run concurrently with the estimation procedure described in Qu, Li, and Lewis (2012) for the computation of the left eigenvector associated with the zero eigenvalue of the Laplacian matrix and for which agents are not required to be aware of their out-neighborhood. To the best of our knowledge, this work introduces the first approach suitable for an implementation based on a pure broadcast communication scheme.

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2. Related works

In this section we review the major contributions available at the state of the art concerning the average consensus problem on digraphs.

In Dominguez-Garcia and Hadjicostis (2011), a doubly stochastic weight matrix is computed by an iterative procedure that adjusts the outgoing weights of each node. Notably, the fact that the columns of the weight matrix sum to one at each step, guarantees that the average consensus can be performed in parallel with respect to the convergence of the weight matrix to a doubly stochastic form.

In Cai and Ishii (2012), the average consensus over a directed network topology is addressed. The proposed algorithms require an augmentation of the variables of each agent adding a "surplus" variable to be sent to the different out-neighbors, thus requiring the knowledge of the out-neighborhood.

In Atrianfar and Haeri (2012), the average consensus problem is addressed both in the continuous time and in the discrete time under the assumption of switching network topology. However, the discrete time consensus algorithm requires the adjacency matrix to be doubly stochastic.

In Hadjicostis and Charalambous (2013), the discrete-time average consensus problem in the presence of bounded delays in the communication links and changing interconnections is addressed. The proposed ratio-consensus protocol requires that each agent is aware of the number of its out-neighborhood.

In Dominguez-Garcia and Hadjicostis (2013), the authors present a class of distributed iterative algorithms to asymptotically scale a primitive column stochastic matrix to a double stochastic and demonstrate the application of these algorithms to the average consensus problem. In particular, each node is in charge of assigning weights on its outgoing edges based on the weights on its incoming edges. Thus, the knowledge of the out-neighborhood is required.

Kempe, Dobra, and Gehrke (2003) propose a gossip-based *push-sum* protocol to compute the average based on the assignment of the weights of the out-going neighbors such that their sum is unitary or, in other terms, the knowledge of each agent's out-degree is required.

Olshevsky and Tsitsiklis (2009) present two different strategies to compute the average when the graph is not balanced. The first one requires the exact knowledge of the left eigenvector whereas the second one assumes bidirectional communications, i.e., an undirected graph. Compared to these algorithms our approach can be run on any strongly connected digraph without any prior knowledge of its left eigenvector.

Consensus in time-varying digraphs is analyzed in Hendrickx and Tsitsiklis (2013) and Touri (2012), giving conditions on the sequence of graphs to ensure convergence to a weighted average of the initial conditions. However, in order to reach the exact average, the sequence of matrices needs to be doubly stochastic or balanced.

Eventually, in Chen, Tron, Terzis, and Vidal (2010) an approach to solve the average consensus on networks with random packet losses is presented. In contrast to our approach, this work requires the agents to send an additional variable keeping track of the changes in the state variables caused by the neighbors influence. However, the assumption on the links failure probabilities implies the existence of bidirectional communications.

3. Preliminaries

Let us consider a set of n agents whose communication network is described by a *digraph* $\mathcal{G}(\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges, i.e., ordered pairs of nodes. Let us define the *weighted adjacency matrix* $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n\times n}$ as follows: $\mathcal{A}_{ij}(\mathcal{G}) > 0$ if $(j, i) \in \mathcal{E}$, $\mathcal{A}_{ij}(\mathcal{G}) = 0$ otherwise.

Note that $\mathcal{A}_{ij}(\mathcal{G}) > 0$ if agent i can receive data from agent j. It is worthwhile to point out that the previously defined adjacency matrix is based on the incoming edges of each node. It is assumed that no self-loops exist in the network, i.e., $(i, i) \notin \mathcal{E}$. The in-degree and the out-degree of a node k are given by $d_{\text{in}}(k) = \sum_j \mathcal{A}_{kj}(\mathcal{G})$ and $d_{\text{out}}(k) = \sum_j \mathcal{A}_{jk}(\mathcal{G})$, respectively. The *Laplacian matrix* is defined as $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, with $\mathcal{D}(\mathcal{G})$ the *diagonal in-degree matrix* defined as $\mathcal{D}(\mathcal{G}) = \left[d_{\text{in}}(1), \ldots, d_{\text{in}}(n)\right]^T$. For the sake of readability, the dependency on the graph $\mathcal{G}$ will be omitted in the rest of the paper. Let us recall that the Laplacian matrix is a non-symmetric weakly diagonal dominant matrix. It has a zero structural eigenvalue for which the corresponding right eigenvector is the vector of ones of appropriate size, i.e., $\mathcal{L}\mathbf{1} = \mathbf{0}$.

Let the following assumptions be satisfied throughout the rest of the paper:

- A1 A unique identifier is associated with each agent i of the network, e.g., the MAC address.
- A2 Each agent sends n variables.
- A3 Each agent does not know the number of agents receiving its information (i.e., its out degree).
- A4 The network topology of the considered multi-agent system is described by a static SCWD.

In A1, we assume that each agent can distinguish the information coming from the other agents according to the identifier of the sender. In A2, it is assumed that each agent has enough storage size for the values coming from its in-neighbors. Therefore, the number of agents belonging to the network is known by each agent. In A3, it is stated that each agent cannot count the number of its out-neighbors. Eventually, in A4 we assume that the information produced by one node is propagated within the network.

4. Decentralized estimation of the left eigenvector

In this section, the distributed procedure for the estimation of the left eigenvector associated with the zero structural eigenvalue of the Laplacian matrix encoding an SCWD proposed in Qu et al. (2012) is briefly reviewed.

Let us consider the Perron matrix $\mathcal{C}$ defined as: $\mathcal{C} = \mathcal{I} - \beta\,\mathcal{L}$ with $0 < \beta < \frac{1}{\Psi}$ and $\Psi = \max_i\{\sum_{j\neq i} \mathcal{A}_{ij}\}$ and let agent i have a variable $\delta_i(k) = [\delta_{i1}(k) \;\cdots\; \delta_{in}(k)]^T$ with initial values $\delta_{ij}(0) = 1$ if $i = j$, 0 otherwise. At each iteration, the agents update their variables as follows:

$$\delta_{ij}(k+1) = \sum_{p\in\mathcal{N}_i\cup i} \mathcal{C}_{ip}\delta_{pj}(k), \tag{1}$$

with $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ the in-neighborhood of agent i. Note that, update rule (1) can be put in vectorial form as: $\Delta(k+1) = \mathcal{C}\,\Delta(k)$, with $\Delta(k) = [\delta_1(k), \ldots, \delta_n(k)]^T$. Noting that $\Delta(0) = \mathcal{I}$, it is easy to see that at iteration k, the variable $\delta_i(k)$ contains exactly the value of the ith row of the matrix $\mathcal{C}^k$.

Let us denote by $\lambda_{\mathcal{C}_i}$ and $\lambda_{\mathcal{L}_i}$, the ith eigenvalue of the Perron matrix $\mathcal{C}$ and of the Laplacian matrix $\mathcal{L}$, respectively, for which holds: $\lambda_{\mathcal{C}_i} = 1 - \beta\,\lambda_{\mathcal{L}_i}$. It follows that the two matrices also share the same set of eigenvectors. In particular for the eigenvalue of maximum modulus of the Perron matrix $\mathcal{C}$, namely $\lambda_{\mathcal{C}_1} = 1$, to which corresponds the zero eigenvalue of the Laplacian matrix $\mathcal{L}$, namely $\lambda_{\mathcal{L}_1} = 0$ we have that $\mathcal{C}\,\mathbf{1} = \lambda_{\mathcal{C}_1}\mathbf{1}$ and $w^T\,\mathcal{C} = \lambda_{\mathcal{C}_1} w^T$, with w^T the left eigenvector associated with $\lambda_{\mathcal{C}_1}$ and $\lambda_{\mathcal{L}_1}$.

From the Perron–Frobenius theorem it follows that if the graph is strongly connected by applying the update rule given in (1), then $\lim_{k\to\infty} \Delta(k) = \frac{\mathbf{1}\,w^T}{w^T\mathbf{1}}$ or, in other terms, $\delta_i(k)$ will tend to the normalized left eigenvector w of the Laplacian matrix encoding the digraph.