Mechanical Systems and Signal Processing ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

Directional algorithms for the frequency isolation problem in undamped vibrational systems

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ARTICLE INFO

Article history:
Received 2 March 2015
Received in revised form
1 December 2015
Accepted 29 December 2015

Keywords:
Frequency isolation
Resonance
Eigenvalue perturbation
Partial pole assignment

ABSTRACT

A new algorithm is presented to solve the frequency isolation problem for vibrational systems with no damping: given an undamped mass-spring system with resonant eigenvalues, the system must be re-designed, finding some close-by non-resonant system at a reasonable cost. Our approach relies on modifying masses and stiffnesses along directions in parameter space which produce a maximal variation in the resonant eigenvalues, provided the non-resonant ones do not undergo large variations. The algorithm is derived from first principles, implemented, and numerically tested. The numerical experiments show that the new algorithms are considerably faster and more robust than previous algorithms solving the same problem.

1. Introduction

It is well known [13] that whenever the natural frequencies of a vibrating structure are close to the frequencies of some external force, these vibrations may be amplified to the point of becoming dangerous. This is the so-called phenomenon of *resonance*. The external forces may be, for instance, those produced by the waves affecting an off-shore oil platform [1, p. 146], an earthquake acting on a building [16,p. xv] or, maybe the most notorious example recently, the steps of pedestrians walking on the London Millenium Bridge [18, p. 235].

To model resonance in mathematical terms, some interval on the real line is typically identified as the *resonance band*, i.e., the region which should be free of natural frequencies in order to guarantee non-resonance. One example of this is the earthquake band prescribed by the California State Building Department: in order to minimize damage, the natural frequencies of any new building constructed in California must be outside that band [16, p. xv].

In this paper we propose new algorithms for a *frequency isolation problem*, which occurs whenever an initial design for a vibrational structure has some of its natural frequencies within the resonance band. In order to avoid resonance, the system must be re-designed in such a way that all new natural frequencies lie outside the resonance band, and this should be done in such a way that the impact (or the cost) of re-design is small, i.e., the new non-resonant structure is close in some sense to the initial one. In our case we focus on undamped mass-spring systems

$$Mx'' + Kx = F(t), \tag{1}$$

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[1] The research of this author was partially supported by the Spanish Ministerio de Economía y Competitividad under Grant MTM2014-54692-P.
[2] The research of this author was partially supported by FONDECYT under Grant FONDECYT 10050026, and by Universidad Católica del Norte internal Grant DGIP-UCN.

http://dx.doi.org/10.1016/j.ymssp.2015.12.030
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which appear frequently in structural engineering problems. The unknown $x = x(t)$ is an n-vector, and the $n \times n$ matrices M and K are

$$M = \text{diag}\,(m_1, \dots, m_n), \quad m_i > 0,\; i = 1, \dots, n, \tag{2}$$

with K symmetric positive definite and tridiagonal. We consider fixed-free boundary conditions,[1] i.e.,

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix} \tag{3}$$

with $k_i > 0,\; i = 1, \dots, n$. This corresponds to a configuration with n masses connected by n springs of stiffnesses $k_i > 0$, attaching the first spring to a wall and leaving the last mass free.

If periodic solutions $x(t) = ue^{i\omega}t$ of the unforced equations $Mx'' + Kx = 0$ are sought, then the vector u and the scalar ω satisfy $(\omega^2 M - K)u = 0$. In other words, the natural frequencies of (1) are the square roots of the eigenvalues, and the natural modes of vibration are the corresponding eigenvectors of the generalized eigenvalue problem

$$(K - \lambda M)u = 0. \tag{4}$$

Sometimes (see e.g. [10]) it is convenient to rewrite this generalized eigenvalue problem as a conventional one

$$Jv = \lambda v,$$

for the matrix

$$J = M^{-1/2} K M^{-1/2} = \begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{\sqrt{m_1 m_2}} & & & \\ -\frac{k_2}{\sqrt{m_1 m_2}} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{\sqrt{m_2 m_3}} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{k_{n-1}}{\sqrt{m_{n-2} m_{n-1}}} & \frac{k_{n-1}+k_n}{m_{n-1}} & -\frac{k_n}{\sqrt{m_{n-1} m_n}} \\ & & & -\frac{k_n}{\sqrt{m_{n-1} m_n}} & \frac{k_n}{m_n} \end{bmatrix}. \tag{5}$$

The natural frequencies are the square roots of the eigenvalues of J, and the natural modes of vibration are $u = M^{-1/2}v$, where v is the corresponding eigenvector. Notice that the matrix J is also symmetric positive definite and tridiagonal.

In this setting, the frequency isolation problem can be posed as follows:

Frequency isolation problem: Given a resonance band $(c-r, c+r) \subset \mathbb{R}$ and matrices $\overline{M}, \overline{K}$ as in (2) and (3) such that some eigenvalues of $(\overline{K} - \lambda \overline{M})u = 0$ lie inside the resonance band, find new matrices M^* and K^*, also as in (2) and (3), and close, respectively, to $\overline{M}$ and $\overline{K}$, such that no eigenvalue of $(K^* - \lambda M^*)u = 0$ lies inside the resonance band.

The frequency isolation problem is conceptually close to other well-known problems in the theory of vibrating systems (and control theory), like the *partial pole placement problem* [5,6] and the *eigenvalue embedding problem* [4] or, more generally, *model updating problems* [12]. In all of them a potentially dangerous subset of the spectrum must be moved elsewhere by appropriately updating the system parameters. However, there are important differences between them: while in the frequency isolation problem no restriction is imposed on the 'non-dangerous' part of the spectrum, in the other three problems this part of the spectrum and the corresponding eigenvectors must remain fixed. In fact, in model updating problems, both a part of the spectrum and the corresponding eigenvectors must be preserved. Also, in the frequency isolation problem, the main concern is to find a reasonably small update, and structure must be preserved, i.e., the update must keep stringent symmetry, bandedness and definiteness constraints on the matrix coefficients of (4).

One comment to be made is that, of course, one can pose a mathematically more demanding question by asking M^* and K^* to be *as close as possible*, for some appropriate metric in space (M,K), to $\overline{M}$ and $\overline{K}$ within the class of matrices of the forms (2) and (3). However, this makes the problem considerably more difficult, since then it becomes an optimization problem *with constraints on interior eigenvalues*. Whereas constraints on the largest (or smallest) eigenvalues can be treated by using, for instance, semidefinite programming, in our case the resonant frequencies can be placed anywhere in the spectrum, and this makes the optimization problem notoriously harder to deal with. Therefore, we will be deliberately vague, not specifying the degree of closeness between (M^*, K^*) and $(\overline{M}, \overline{K})$. As a matter of fact, in practice, the decision on what is close and what is not depends ultimately on the real-world constraints on the particular application which (4) comes from.

The frequency isolation problem has been previously addressed by Joseph [14], who proposed a Newton-type method for structures vibrating at low frequencies. The computational cost of this algorithm, however, was very high, and the reconstructed system was frequently far away from the initial one. Later, Egaña et al. proposed in [10] a less costly inverse eigenvalue method: a target spectrum away from the resonance band is fixed in advance. Although there is an infinity of

[1] We focus on these boundary conditions for the sake of concretion, but other boundary conditions allow for a similar treatment.