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Technical communique

A multivariable super-twisting sliding mode approach☆

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ABSTRACT

This communique proposes a multivariable super-twisting sliding mode structure which represents an extension of the well-known single input case. A Lyapunov approach is used to show finite time stability for the system in the presence of a class of uncertainty. This structure is used to create a sliding mode observer to detect and isolate faults for a satellite system.

1. Introduction

Sliding mode control has been an active area of research for many decades due its (at least theoretical) invariance to a class of uncertainty known as matched uncertainty (Utkin, 1992). More recently these ideas have been exploited extensively for the development of robust observers and have found applications in the area of fault detection and fault tolerant control (Alwi, Edwards, & Tan, 2011; Fridman, Davila, & Levant, 2008). However one of the disadvantages of traditional sliding mode control (1st order sliding modes) is the 'chattering' due to the discontinuous control action (Utkin, 1992). Higher order sliding modes (HOSM) remove the chattering effect while retaining the robustness of first order sliding modes and improving on their accuracy (Fridman & Levant, 1996; Levant, 1993). A disadvantage of imposing an r-th order sliding mode is the necessity of having $s, \dot{s}..s^{r-1}$ available (where $s(t)$ is the switching surface). However in one special case of second order sliding modes, the derivative information is not required. This is the so-called 'super-twisting' approach (Levant, 1998). Until very recently stability, robustness and convergence rates in higher order sliding mode methods have been analyzed in terms of homogeneity or geometric arguments (Levant, 2005). However in a succession of papers (Moreno & Osorio, 2008, 2012; Polyakov & Poznyak, 2009), Lyapunov methods were employed successfully for the first time to analyze the properties of the super-twisting algorithm for uncertain systems. This has opened the door for the integration of these ideas with other nonlinear tools including gain adaptation (Alwi & Edwards, 2013; Gonzalez, Moreno, & Fridman, 2012; Shtessel, Moreno, Plestan, Fridman, & Poznyak, 2010). However in all these developments a single input control structure has essentially been considered. In many situations it is possible by control input scaling to transform a multi-input control problem with m control inputs into a decoupled problem involving m single input control structures and so the approaches in Alwi and Edwards (2013), Gonzalez et al. (2012) and Shtessel et al. (2010) work satisfactorily. Instead, in this communique, a multivariable super-twisting structure is proposed, which is then analyzed using an extension of the Lyapunov ideas from Moreno and Osorio (2012). An example involving a fault detection problem in a satellite system is used to demonstrate a situation in which the proposed multi-input super-twisting structure is useful. The notation used in the paper is quite standard—in particular, throughout the paper, $\|\cdot\|$ is used to represent the Euclidean norm.

2. Problem statement and system description

In multivariable sliding mode control and observation, the objective is to force to zero in finite time a constraint (or switching) function given by $\sigma(x)$, where $x \in \mathbb{R}^n$ is the state of the dynamical system and $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^m$ Shtessel, Edwards, Fridman, and Levant (2013). In calculating the total time derivative of σ, for the case of

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conventional (first order) sliding modes, an expression

$$\dot{\sigma}(t) = a(t,x) + b(t,x)v + \gamma(t,\sigma) \tag{1}$$

is established where v is the manipulated variable (the control signal or the output error injection in the case of observer problems), $a(t,x) \in \mathbb{R}^m$ and $b(t,x) \in \mathbb{R}^{m\times m}$ are assumed to be known, and $\gamma(\cdot)$ represents unknown (but usually bounded) uncertainty. If $\det(b(t,x)) \neq 0$ then using the expression $v = b(t,x)^{-1}(\bar{v} - a(t,x))$ where the components of $\bar{v}$ are

$$\bar{v}_i = -k_1 \mathrm{sign}(\sigma_i)|\sigma_i|^{1/2} - k_2\sigma_i + z_i \tag{2}$$

$$\dot{z}_i = -k_3 \mathrm{sign}(\sigma_i) - k_4\sigma_i \tag{3}$$

and $k_1, \ldots, k_4$ are scalar gains, the system

$$\dot{\sigma}_i = -k_1 \mathrm{sign}(\sigma_i)|\sigma_i|^{1/2} - k_2\sigma_i + z_i + \gamma_i(t,\sigma) \tag{4}$$

$$\dot{z}_i = -k_3 \mathrm{sign}(\sigma_i) - k_4\sigma_i \tag{5}$$

for $i = 1 \ldots m$ is obtained. Suppose $|\gamma_i(t,\sigma)| \leq d_i|\sigma_i|$ for some scalars d_i, then if the gains $k_1 \ldots k_4$ are chosen properly, it can be proved that $\sigma_i = \dot{\sigma}_i = 0$ in finite time: see for example Moreno and Osorio (2012). Alternatively if $|\dot{\gamma}_i(t,\sigma)| \leq \bar{d}_i$ for some finite gains $\bar{d}_i$, then for appropriate gains $k_1 \ldots k_4$, it can be proved that $\sigma_i = \dot{\sigma}_i = 0$ in finite time: see Levant (1993) and Moreno and Osorio (2012). In the literature such a controller is usually known as a super-twisting controller (Fridman & Levant, 1996; Levant, 1993, 1998).

Suppose instead of (2)–(3) a non-decoupled injection term

$$\bar{v} = -k_1 \frac{\sigma}{\|\sigma\|^{1/2}} + z - k_2\sigma \tag{6}$$

$$\dot{z} = -k_3 \frac{\sigma}{\|\sigma\|} - k_4\sigma \tag{7}$$

is used where $k_1, \ldots, k_4$ are scalars. Then the result is a set of coupled equations rather than the decoupled structure in (4)–(5), and the work in Moreno and Osorio (2012) cannot be employed directly. (Note however, if $m = 1$ then the scalar control structure in (6)–(7) reverts to (2)–(3). Also in this situation $k_2 = k_4 = 0$ is usually selected.) Substituting (6) into (1) yields a special case of the system

$$\dot{\sigma} = -k_1 \frac{\sigma}{\|\sigma\|^{1/2}} + z - k_2\sigma + \gamma(t,\sigma) \tag{8}$$

$$\dot{z} = -k_3 \frac{\sigma}{\|\sigma\|} - k_4\sigma + \phi(t) \tag{9}$$

when $\phi(t) \equiv 0$. The term $\phi(t)$ in (9) is included here to maintain compatibility with the more generic formulation in Moreno and Osorio (2012), and will be exploited in the example in Section 3. The terms $\gamma(t,\sigma)$ and $\phi(t)$ are assumed to satisfy

$$\|\gamma(t,\sigma)\| \leq \delta_1\|\sigma\| \tag{10}$$

$$\|\phi(t)\| \leq \delta_2 \tag{11}$$

for known scalar bounds $\delta_1, \delta_2 > 0$.

Remark 1. Note that the uncertainty classes discussed earlier are a subset of the uncertainty in (10). Also note the matrix $b(t,x)$ must be known to achieve the structures in (8)–(9) (and also the decoupled one in (2)–(3)).

Remark 2. Also note that the differential equations in (4)–(5) and (8)–(9) have discontinuous right hand sides. The solutions to such equations must therefore be understood in the Filippov sense (Filippov, 1998).

Remark 3. Equations such as (8)–(9) can also appear in the context of observer problems as will be demonstrated in Section 3.

Proposition 1. *For the system in (8)–(9), there exist a range of values for the gains $k_1 \ldots k_4$, such that the variables σ and $\dot{\sigma}$ are forced to zero in finite time and remain zero for all subsequent time.*

Proof. For the system (8)–(9), consider as a Lyapunov-function[2] candidate

$$V(\sigma,z) = 2k_3\|\sigma\| + k_4\sigma^T\sigma + \frac{1}{2}z^Tz + \zeta^T\zeta \tag{12}$$

where $\zeta := k_1\frac{\sigma}{\|\sigma\|^{1/2}} + k_2\sigma - z$. Define the subspace

$$\mathcal{S} = \{(\sigma,z) \in \mathbb{R}^{2m} : \sigma = 0\} \tag{13}$$

then $V(\sigma,z)$ in (12) is everywhere continuous, and differentiable everywhere except on the subspace $\mathcal{S}$. Furthermore it is easy to verify that $V(\cdot)$ is positive definite and radially unbounded.

Differentiating the expression in (12) yields

$$\begin{aligned}\dot{V}(\sigma,z) = {} & \left(2k_3 + \frac{k_1^2}{2}\right)\frac{\sigma^T\dot{\sigma}}{\|\sigma\|} + 2\left(\frac{k_2^2}{2} + k_4\right)\sigma^T\dot{\sigma} + 2z^T\dot{z} \\ & + \frac{3}{2}k_1k_2\frac{\sigma^T\dot{\sigma}}{\|\sigma\|^{1/2}} - k_2(\dot{\sigma}^Tz + \sigma^T\dot{z}) \\ & - k_1\left(-\frac{1}{2}\frac{(\sigma^T\dot{\sigma})(z^T\sigma)}{\|\sigma\|^{5/2}} + \frac{(\dot{z}^T\sigma + z^T\dot{\sigma})}{\|\sigma\|^{1/2}}\right)\end{aligned} \tag{14}$$

then substituting for (8)–(9) it follows from (14) using straightforward algebra that

$$\begin{aligned}\dot{V}(\sigma,z) = {} & -\left(k_1k_3 + \frac{k_1^3}{2}\right)\frac{\|\sigma\|^2}{\|\sigma\|^{3/2}} + \frac{3}{2}k_1k_2\frac{\sigma^T\gamma}{\|\sigma\|^{1/2}} \\ & - (k_2k_4 + k_2^3)\|\sigma\|^2 - \left(k_4k_1 + \frac{5}{2}k_1k_2^2\right)\frac{\|\sigma\|^2}{\|\sigma\|^{1/2}} \\ & + k_1^2\frac{\sigma^Tz}{\|\sigma\|} + 2k_2^2\sigma^Tz + 3k_1k_2\frac{\sigma^Tz}{\|\sigma\|^{1/2}} \\ & - k_2\|z\|^2 + \frac{k_1}{2}\frac{(\sigma^Tz)(z^T\sigma)}{\|\sigma\|^{5/2}} - k_1\frac{z^Tz}{\|\sigma\|^{1/2}} \\ & + \left(2k_3 + \frac{k_1^2}{2}\right)\frac{\sigma^T\gamma}{\|\sigma\|} + (2k_4 + k_2^2)\sigma^T\gamma \\ & - (k_3k_2 + 2k_1^2k_2)\frac{\|\sigma\|^2}{\|\sigma\|} - k_2\gamma^Tz + \frac{k_1}{2}\frac{\sigma^T\gamma z^T\sigma}{\|\sigma\|^{5/2}} \\ & - k_1\frac{z^T\gamma}{\|\sigma\|^{1/2}} + 2z^T\phi - k_2\sigma^T\phi - k_1\frac{\phi^T\sigma}{\|\sigma\|^{1/2}}\end{aligned} \tag{15}$$

for all $(\sigma,z) \notin \mathcal{S}$. Then from simple bounding arguments

$$\begin{aligned}\dot{V}(\sigma,z) \leq {} & -\left(k_1k_3 + \frac{k_1^3}{2}\right)\|\sigma\|^{1/2} - (k_3k_2 + 2k_1^2k_2)\|\sigma\| \\ & - (k_2k_4 + k_2^3)\|\sigma\|^2 - \left(k_4k_1 + \frac{5}{2}k_1k_2^2\right)\|\sigma\|^{3/2} \\ & + k_1^2\frac{|\sigma^Tz|}{\|\sigma\|} + 2k_2^2|\sigma^Tz| + 3k_1k_2\frac{|\sigma^Tz|}{\|\sigma\|^{1/2}} \\ & - k_2\|z\|^2 + \frac{k_1}{2}\frac{|\sigma^Tz|^2}{\|\sigma\|^{5/2}} + \left(2k_3 + \frac{k_1^2}{2}\right)\frac{|\sigma^T\gamma|}{\|\sigma\|} \\ & + (2k_4 + k_2^2)|\sigma^T\gamma| + \frac{3}{2}k_1k_2\frac{|\sigma^T\gamma|}{\|\sigma\|^{1/2}} \\ & + k_2|\gamma^Tz| + \frac{k_1}{2}\frac{|\sigma^T\gamma||z^T\sigma|}{\|\sigma\|^{5/2}} + k_1\frac{|z^T\gamma|}{\|\sigma\|^{1/2}} \\ & + 2z^T\phi + k_2|\sigma^T\phi| + k_1\frac{|\phi^T\sigma|}{\|\sigma\|^{1/2}}.\end{aligned} \tag{16}$$

[2] Note that in the special case when $m = 1$, the Lyapunov function in (12) becomes the one originally proposed in Moreno and Osorio (2012).