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Metrics of performance for discrete-time descriptor jump linear systems☆

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ABSTRACT

Analytical tools to measure the performance of a control system described by a discrete-time descriptor jump linear system are given. Specifically, a closed-form expression for the steady-state output power as well as a bound for the performance index related to the H_∞ control problem are given. The analysis is made by introducing new operators to handle the singularity of the system.

1. Introduction

Closed-form formulas to measure the degradation of performance of a Markov jump linear system (MJLS), which is a non-singular system, have been proposed in Gray, Zhang, and Gonzalez (2003); Ling and Lemmon (2004); Zhang, Gray, and González (2008); Wang, Gray, González, and Chávez (2013).

These available tools cannot be adapted in a simple way to handle systems with singularities. In this paper, a closed-form expression for the steady-state output power to evaluate the performance of a control singular system as well as a bound for the norm of the H_∞ control problem are given.

Singular systems with jumps are also called descriptor jump linear systems (DJLS). In contrast with an MJLS, the difficulty of dealing with a DJLS lies in the presence of a singular factor on the left hand side of the state equation of the system. Recursive equations that play a central role in MJLS cannot be directly employed here; instead, a more fruitful approach is to consider algebraic equations like (5), which is handled by introducing the new operator $\mathcal{T}$ (see (3)), and Eqs. (7) and (11) which are handled by introducing the perturbed operator $\mathcal{T}_\epsilon$ (see (8)).

The formula for the steady-state output power, denoted by J_s, is derived in Section 2. A bound for the index of performance related to the H_∞ control problem, which is denoted by J_∞, is obtained in Section 3. The conclusions are given in Section 4.

2. The steady-state output power

Consider the following DJLS:

$$S_{\theta(k+1)}\boldsymbol{x}(k+1) = A_{\theta(k)}\boldsymbol{x}(k) + B_{\theta(k)}\boldsymbol{\omega}(k) \tag{1a}$$

$$\boldsymbol{y}(k) = C_{\theta(k)}\boldsymbol{x}(k), \tag{1b}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state continuous vector, $\boldsymbol{\omega}(k) \in \mathbb{R}^q$ is a disturbance input and $\boldsymbol{y}(k) \in \mathbb{R}^p$ is the output of the system. The nonnegative integer variable $k = 0, 1, \ldots$ denotes the sample period number. The initial state $x(0) = \boldsymbol{x}_0$ ($\boldsymbol{x}_0 \in X \subset \mathbb{R}^n$) is a random vector with finite second moment. The process $\boldsymbol{\theta}(k)$ is a first-order homogeneous Markov chain (MC) taking values in $S_\theta \triangleq \{1, \ldots, N\}$, $N \geq 2$. The transition probability matrix of $\boldsymbol{\theta}(k)$ is denoted by $P = [p_{ij}]$, $i, j \in S_\theta$, and $\pi(k) = [p_1(k) \ \ldots \ p_N(k)]$ denotes the kth state probability vector, where $p_i(k) \triangleq \Pr(\boldsymbol{\theta}(k) = i)$, $k \geq 0$. In particular when $k = 0$, we denote $\pi(0) = \pi \in \Theta$, where Θ denotes the set of all initial distributions of $\boldsymbol{\theta}(k)$. It is assumed that $\boldsymbol{\theta}(k)$ is ergodic so that there exists p_i (irrespective of π) such that $p_i = \lim_{k\to\infty} p_i(k)$. It is assumed that $\boldsymbol{\theta}(0)$ and $\boldsymbol{x}(0)$ are independent. For $i \in S_\theta$, the matrices A_i, B_i, and C_i have appropriate dimensions, S_i is an $n \times n$ matrix with $\operatorname{rank}(S_i) = r_i \leq n$. When $r_i = n$ for all $i \in S_\theta$ the system is called non-singular; otherwise it is called singular. For the remainder of the paper, $\boldsymbol{x}(k)$ is assumed to be a well-defined second order random variable (e.g., ensuring that $E\{\boldsymbol{x}(k)\boldsymbol{x}'(k)\}$ exists).

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The symbol $\|\cdot\|$ denotes the Euclidean norm and $[\ \cdot\ ]'$ stands for the matrix transpose.

Definition 1. Consider System (1). Then

$$J_s \triangleq \lim_{k\to\infty} E\{\|\boldsymbol{y}(k)\|^2\},$$

$$J_\infty \triangleq \sup_{\theta(0)\in\Theta,\boldsymbol{w}\in\ell_2} \frac{\sum_{k=0}^{\infty} E\{\|\boldsymbol{y}(k)\|^2\}}{\sum_{k=0}^{\infty} E\{\|\boldsymbol{w}(k)\|^2\}}, \quad \boldsymbol{x}_0 = 0.$$

Note that $\|\boldsymbol{\omega}\|^2 \triangleq \sum_{k=0}^{\infty} E\{\|\boldsymbol{w}(k)\|^2\}$. Following (Costa, Fragoso, & Marques, 2005) let us define

$$Q(k) \triangleq E\left\{\boldsymbol{x}(k)\boldsymbol{x}'(k)\right\}, \tag{2a}$$

$$Q_i(k) \triangleq E\left\{\boldsymbol{x}(k)\boldsymbol{x}'(k)1_{\{\theta(k)=i\}}\right\}, \quad i \in S_\theta, \tag{2b}$$

$$W_i(k) \triangleq E\left\{\boldsymbol{\omega}(k)\boldsymbol{\omega}'(k)1_{\{\theta(k)=i\}}\right\}, \quad i \in S_\theta \tag{2c}$$

and the following bounded operators:

$$\mathcal{A} \triangleq \left(P' \otimes I_{n^2}\right) \operatorname{diag}\left(A_1 \otimes A_1, \dots, A_N \otimes A_N\right),$$
$$\mathcal{B} \triangleq \left(P' \otimes I_{n^2}\right) \operatorname{diag}\left(B_1 \otimes B_1, \dots, B_N \otimes B_N\right),$$
$$\mathcal{S} \triangleq \operatorname{diag}\left(S_1 \otimes S_1, \dots, S_N \otimes S_N\right),$$
$$\mathcal{T} \triangleq \mathcal{S} - \mathcal{A}. \tag{3}$$

Definition 2 is adapted from Costa et al. (2005); Costa and Marquez (1998). If a symmetric matrix A is positive semi-definite it is written as $A \geq 0$.

Definition 2. System (1a) is said to be mean square stable (MSS) if for any $\boldsymbol{x}_0 \in X$ and any $\pi \in \Theta$ there exists a matrix $Q \geq 0$ such that $Q = \lim_{k\to\infty} Q(k)$.

The following lemma, given in Chávez, Costa, and Terra (2011), characterizes the MSS of System (1a) in terms of the matrices $Q_i(k)$.

Lemma 1. *System (1a) is MSS if and only if for any $\boldsymbol{x}_0 \in X$ and any $\pi \in \Theta$ there exists $Q_i \geq 0$ such that $Q_i = \lim_{k\to\infty} Q_i(k)$ for all $i \in S_\theta$.*

The matrix Q_i can be obtained from Lemma 2, whenever $\mathcal{T}$ is invertible. To present this result, let us write $Q = \left[\widetilde{Q}_1' \cdots \widetilde{Q}_N'\right]'$, where $\widetilde{Q}_i = \operatorname{vec}(Q_i)$, $i \in S_\theta$.

Lemma 2. *Let System (1a) be MSS and assume that $\boldsymbol{\omega}(k)$ is a zero mean white noise process with identity covariance matrix I_q and independent of $\boldsymbol{\theta}(k)$ and $\boldsymbol{x}_0$. Then the following equation holds:*

$$\mathcal{T} Q = \mathcal{B} p, \tag{4}$$

where $p = \left[\widetilde{p}_1' \dots \widetilde{p}_N'\right]'$ and $\widetilde{p}_i = \operatorname{vec}(I_q)p_i$, $i \in S_\theta$.

Proof. Fix $j \in S_\theta$. Since $E\{\boldsymbol{\omega}(k)\boldsymbol{\omega}'(\ell)\} = I_q 1_{\{k=\ell\}}$, then

$$\begin{aligned} & S_j E\left\{\boldsymbol{x}(k+1)\boldsymbol{x}'(k+1)1_{\{\theta(k+1)=j\}}\right\} S_j' \\ &= \sum_{i=1}^{N} \left[A_i E\left\{\boldsymbol{x}(k)\boldsymbol{x}'(k)1_{\{\theta(k+1)=j\}}1_{\{\theta(k)=i\}}\right\} A_i'\right] \\ &\quad + \sum_{i=1}^{N} \left[A_i E\left\{\boldsymbol{x}(k)\boldsymbol{\omega}'(k)1_{\{\theta(k+1)=j\}}1_{\{\theta(k)=i\}}\right\} B_i'\right] \\ &\quad + \sum_{i=1}^{N} \left[B_i E\left\{\boldsymbol{\omega}(k)\boldsymbol{x}'(k)1_{\{\theta(k+1)=j\}}1_{\{\theta(k)=i\}}\right\} A_i' \right.\\ &\quad \left. + B_i E\left\{I_q 1_{\{\theta(k+1)=j\}}1_{\{\theta(k)=i\}}\right\} B_i'\right]. \end{aligned}$$

Since $\boldsymbol{\omega}(k)$ is a zero mean white noise and taking into account the independence assumption this equation can be reduced to

$$\begin{aligned} & S_j E\left\{\boldsymbol{x}(k+1)\boldsymbol{x}'(k+1)1_{\{\theta(k+1)=j\}}\right\} S_j' \\ &= \sum_{i=1}^{N} \Big[A_i E\left\{\boldsymbol{x}(k)\boldsymbol{x}'(k)1_{\{\theta(k+1)=j\}}1_{\{\theta(k)=i\}}\right\} A_i' \\ &\quad + B_i E\left\{I_q 1_{\{\theta(k+1)=j\}}1_{\{\theta(k)=i\}}\right\} B_i'\Big] \end{aligned}$$

and due to the fact that $\boldsymbol{\theta}(k)$ is an MC this can be written as follows:

$$\begin{aligned} & S_j E\left\{\boldsymbol{x}(k+1)\boldsymbol{x}'(k+1)1_{\{\theta(k+1)=j\}}\right\} S_j' \\ &= \sum_{i=1}^{N} [A_i E\{\boldsymbol{x}(k)x'(k)1_{\{\boldsymbol{\theta}(k)=i\}}\}p_{ij}A_i' + B_i I_q p_{ij} p_i(k) B_i'] \\ &= \sum_{i=1}^{N} \left[A_i Q_i(k) p_{ij} A_i' + B_i I_q p_{ij} p_i(k) B_i'\right]. \end{aligned}$$

Hence,

$$S_j Q_j(k+1) S_j' - \sum_{i=1}^{N} A_i Q_i(k) A_i' p_{ij} = \sum_{i=1}^{N} B_i I_q B_i' p_{ij} p_i(k).$$

Lemma 1 and the ergodicity of $\boldsymbol{\theta}(k)$ make it possible to take limits as $k \to \infty$ on both sides of this equation resulting in

$$S_j Q_j S_j' - \sum_{i=1}^{N} A_i Q_i A_i' p_{ij} = \sum_{i=1}^{N} B_i I_q B_i' p_{ij} p_i. \tag{5}$$

The claim follows by applying the vec operator in (5) and collecting all values of $j \in S_\theta$. $\square$

The column vector Q has N column-block vectors. Since $\widetilde{Q}_i = \operatorname{vec}(Q_i)$, $Q_i = \operatorname{vec}^{-1}\left(\widetilde{Q}_i\right)$, where $\widetilde{Q}_i$ is the ith block of Q, which can be calculated from Eq. (4), whenever $\mathcal{T}$ is invertible. Now a closed-form expression for J_s can be obtained.

Theorem 1. *Let System (1) be MSS and let $\mathcal{T}$ be invertible. Then*

$$J_s = \sum_{i=1}^{N} \operatorname{tr}\left(C_i Q_i C_i'\right). \tag{6}$$

Proof. From (1b) and (2b), it follows

$$\begin{aligned} E\left\{\|\boldsymbol{y}(k)\|^2\right\} &= E\left\{\boldsymbol{x}'(k)C_{\theta(k)}'C_{\theta(k)}\boldsymbol{x}(k)\right\} \\ &= E\left\{\operatorname{tr}\left(C_{\theta(k)}\boldsymbol{x}(k)\boldsymbol{x}'(k)C_{\theta(k)}'\right)\right\} \\ &= E\left\{\operatorname{tr}\left(\sum_{i=1}^{N} C_i\boldsymbol{x}(k)\boldsymbol{x}'(k)1_{\{\theta(k)=i\}}C_i'\right)\right\} \\ &= \sum_{i=1}^{N} \operatorname{tr}\left(C_i Q_i(k) C_i'\right). \end{aligned}$$

Since the systems is MSS, Eq. (6) follows from Lemmas 1 and 2, by taking limits as $k \to \infty$ on both sides of this equation. $\square$

Remark. According to Lemma 2 if we take the set of matrices $(1+\epsilon^2)^{-1/2}A_i$ and $p_i^{-1/2}B_i^{1/2}$ for a small enough $\epsilon > 0$ such that $(\mathcal{T} - \epsilon^2\mathcal{A})$ is invertible, then there exists a unique set of matrices $Q_j \geq 0$ satisfying the system

$$S_j Q_j S_j' - (1+\epsilon^2)\sum_{i=1}^{N} A_i Q_i A_i' p_{ij} = \sum_{i=1}^{N} B_i p_{ij}. \tag{7}$$

This fact will be used in the following section.