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Time cardinality constrained mean–variance dynamic portfolio selection and market timing: A stochastic control approach☆

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ABSTRACT

An investor does not always invest in risky assets in all the time periods, often due to a market timing consideration and various forms of market friction, including the management fee. Motivated by this observed common phenomenon, this paper considers the time cardinality constrained mean–variance dynamic portfolio selection problem (TCCMV) in markets with correlated returns and in which the number of time periods to invest in risky assets is limited. Both the analytical optimal portfolio policy and the analytical expression of the efficient mean–variance (MV) frontier are derived for TCCMV. It is interesting to note whether investing in risky assets or not in a given time period depends entirely on the realization of the two adaptive processes which are closely related to the local optimizer of the conditional Sharpe ratio. By implementing such a solution procedure for different cardinalities, the MV dynamic portfolio selection problem with management fees can be efficiently solved for a purpose of developing the best market timing strategy. The final product of our solution framework is to provide investors advice on the best market timing strategy including the best time cardinality and its distribution, as well as the corresponding investment policy, when balancing the consideration of market opportunity and market frictions.

The ground-breaking mean–variance (MV) formulation proposed by Markowitz (1952) 60 years ago initialized the modern era of portfolio selection. The earliest work on dynamic portfolio selection was dominated by the utility maximization framework pioneered by Merton (1969). After the static MV portfolio selection theory was extended to continuous-time MV portfolio selection by Bajeux-Besnainou and Portait (1998) using the martingale approach, and to multi-period and continuous-time MV portfolio selection, respectively, by Li and Ng (2000) and Zhou and Li (2000) using the stochastic control approach, the past decade has witnessed significant advancement of both theory and methodologies for dynamic MV portfolio selection by leaps and bounds, see, for example, Basak and Chabakauri (2010), Bielecki, Jin, Pliska, and Zhou (2005), Cui, Gao, Li, and Li (2014), Cui, Li, Wang, and Zhu (2012), Li, Zhou, and Lim (2001), Lim and Zhou (2002) and Zhu, Li, and Wang (2004).

An investor does not always invest in risky assets in all time periods, often due to a market timing consideration and various forms of market frictions, including management fees. Motivated by this observed common phenomenon, we investigate in this paper the time cardinality constrained mean–variance dynamic portfolio selection problem (TCCMV) in markets with correlated returns and in which the number of time periods to invest in risky assets is limited, and, furthermore, the mean–variance dynamic portfolio selection problem with management fees for a purpose of developing better market timing strategies.

Market timing strategies (MTS), which have been widely adopted in financial practice, refer to investment strategies that *strategically* shift the fund completely between risky and riskfree assets after observing and predicting market conditions. Based on one- or two-step predictions of the expected return of the stocks,

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Pesaran and Timmermann (1994) studied how investors should shift their funds between risky and riskfree accounts and proposed implementing MTS in a fashion of rolling horizon: choose from among four possible choices at each time t for the two periods ahead (invest or do not invest in risky assets in both $t+1$ and $t+2$, or choose one period between $t+1$ and $t+2$ to invest). Breen, Glosten, and Jagannathan (1989) and Pesaran and Timmermann (1994) carried out empirical studies which verified that MTS outperforms buy-and-hold policies due to utilization of its prediction power. Considering the transaction cost, Li and Lam (2002) investigated the optimal market timing strategy by maximizing the expected return. Recently, Schizas and Thomakos (2015) carried on an empirical study on viability of a market timing strategy based on the pair-trading. Readers who are interested in this subject may refer to the references in Schizas and Thomakos (2015).

Investors gain useful information for the future from the past market movement (thus possessing certain kind of prediction power) only when the returns are correlated among different time periods. While most of the studies in discrete-time portfolio selection assume time independency of the return vector, exceptions include Çakmak and Özekici (2006), Costa and Araujo (2008) and Costa and Oliverira (2012) where a Markovian process is adopted to model the dynamics of the price process. General forms of correlation structure are assumed for returns in the portfolio selection formulations of Dokuchaev (2007, 2012), Xu and Li (2008) and Zhang and Li (2012). Černý and Kallsen (2009) studied the variance-optimal hedging problem in a framework under general semi-martingale price process, which includes both discrete-time and continuous-time MV models as its special cases and allows the returns to be correlated.

Existence of market frictions is the driving force behind the emergence of MTS. Market frictions constitute the cost of investment, thus a hidden reason that prevents investors from holding the risky assets all the time. In plain language, MTS is to identify the timing when the benefits from investment overcome the cost. One typical form of market friction involves management fees charged by fund managers for managing the portfolio. Take the practice of AIS (The American Investment Service, 2010) as an illustration: For investors with assets under management (AUM) between US\$100,000 and US\$250,000, the annual fee charged by AIS is 0.80% of AUM or US\$1500, whichever is greater. Due to the set-up type of management costs for managing the risky assets, investors do not always invest in risky assets in all time periods. Dokuchaev (2012) considered another type of management fee, i.e., the proportional management cost, in his study of portfolio allocation under a framework of utility maximization.

The contributions of this paper are two-fold, in both building up model formulations to address a real-world challenge in market timing under a mean–variance framework and deriving corresponding analytical solutions. For the first time in the literature, we propose a formal model of TCCMV for market timing under a multi-period mean–variance framework, while the current MTS literature only adopts the expected portfolio return as the indicator to decide the best timing and has been dominated by the results for short time periods, two time periods for almost all the cases. Furthermore, we propose a novel model for mean–variance dynamic portfolio selection with management fees (of a set-up type) which yields the optimal market timing strategy including the best time cardinality and its distribution, as well as the corresponding portfolio policy, such that achieving a balance between market opportunity and market friction. Technically, solving our proposed new model for TCCMV and MV portfolio selection with set-up type management fees invokes innovative stochastic control approach with a distinct feature of cardinality constraints on control, which has been only developed recently for *deterministic* cardinality constrained linear–quadratic control (Gao & Li, 2011). Our stochastic model setting demands new efforts to establish a solution framework for *stochastic* cardinality constrained linear–quadratic control. The consideration of market timing and the set-up type of management fee essentially make the control mode switching between the two: Investing in risky assets or withdrawing completely from risky assets. The stochastic TCCMV problem studied in this paper requires exploration of *sufficient statistics* extracted from market history and correlation structure of the returns to build up a feedback control law to handle both discrete and continuous types of control, corresponding to the best investment timing and optimal portfolio weights, respectively.

The remaining of this paper is organized as follows. We first give the problem formulation for TCCMV in Section 1. We then derive in Section 2 the optimal portfolio policies for TCCMV with its financial implications. We discuss further in Section 3 mean–variance portfolio optimization with management fee of a set-up type, which offers advice to investors on the best market timing strategy including the best time cardinality and its distribution, as well as the corresponding investment policy. We present in Section 4 an example to illustrate the proposed solution procedure and the derived best market timing strategy. Finally, we conclude the paper in Section 5. We use $\pi(\cdot)$ and $v(\cdot)$ to denote the optimal control (policy) and the optimal value of problem $(\cdot)$, use $\mathbf{1}$ for a vector with all elements being 1 and denote by $A \succ 0$ for a positive definite matrix A.

1. Problem formulation for time cardinality constrained market timing

We consider a capital market with n risky assets and one riskless asset, all of which evolve within a time horizon of T periods. Let the returns of the riskless asset be $r_t,\ t=0,\ldots,T-1$, which are assumed to be deterministic. Denote the return vector of risk assets in period t by $\mathbf{e}_t \triangleq \left(e_t^1, e_t^2, \ldots, e_t^n\right)'$, which is a square integrable random vector. In our study, we allow return vectors in different time periods, $\{\mathbf{e}_t\}|_{t=0}^{T-1}$, to be statistically correlated. All the randomness is modeled by a standard probability space $\{\Omega, \{\mathcal{F}_t\}|_{t=0}^{T}, \mathbb{P}\}$, where Ω is the event set, $\mathcal{F}_t$ is the σ-algebra of the event available at time t with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathbb{P}$ is the probability measure. Let u_t^i be the dollar amount invested in the ith risky asset in period t, $i=1,\ldots,n$, and x_t be the wealth level in period t. Then, given the initial wealth x_0, the dynamics of wealth evolves as follows,

$$\begin{aligned} x_{t+1} &= r_t(x_t - \mathbf{1}'\mathbf{u}_t) + \mathbf{e}_t'\mathbf{u}_t \\ &= r_t x_t + \mathbf{P}_t'\mathbf{u}_t, \quad t=0,\ldots,T-1, \end{aligned} \tag{1}$$

where $\mathbf{u}_t \triangleq \left(u_t^1, u_t^2, \ldots, u_t^n\right)'$ and $\mathbf{P}_t \triangleq \left(P_t^1, P_t^2, \ldots, P_t^n\right)' = \mathbf{e}_t - r_t\mathbf{1}$ is the excess return vector. We say that $\mathbf{u}_t$ is an admissible trading strategy if it is $\mathcal{F}_t$ measurable. We use the notations $\mathrm{E}_t[\cdot]$, $\mathrm{Cov}_t[\cdot]$ and $\mathrm{Var}_t[\cdot]$ to denote the conditional expectation $\mathrm{E}[\cdot|\mathcal{F}_t]$, the conditional covariance matrix $\mathrm{Cov}[\cdot|\mathcal{F}_t]$ and the conditional variance $\mathrm{Var}[\cdot|\mathcal{F}_t]$, respectively. At $t=0$, we simply use $\mathrm{E}[\cdot]$ for $\mathrm{E}[\cdot|\mathcal{F}_0]$. We assume that the conditional covariance matrices $\mathrm{Cov}_t[\mathbf{e}_t]$, $t=0,\ldots,T-1$, are all positive definite (please refer to Cui et al., 2014 and Li et al., 2001).

Let us consider to impose the following *time cardinality constraint* on admissible trading strategies,

$$\sum_{t=0}^{T-1} \delta(\mathbf{u}_t) \le s \le T, \tag{2}$$

where $\delta(\cdot)$ is the indicator function, i.e., $\delta(\mathbf{a}) = 0$ if $\mathbf{a}$ is a zero vector and $\delta(\mathbf{a}) = 1$ otherwise, with a purpose to identify the best s periods out of the T time periods for investment. We term the following problem as the *time cardinality* constrained MV portfolio