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Synchronization control in networks with uniform and distributed phase lag[☆]

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ABSTRACT

We show that phase lag angles in oscillator networks can be used to control the frequency of synchronized oscillations, either to adjust the common frequency to any preset value, within limits, or else to damp out any highly oscillatory nodes so that the system oscillates almost independently of the phase lag. We investigate in particular the Sakaguchi–Kuramoto model and a generalization with nonisochronous oscillations, for globally connected networks, to show that synchronization occurs under a broad set of conditions for both uniform and distributed phase lag, and find specific formulas for the common frequency of oscillation. The analysis is valid for any finite number of nodes and for arbitrary distributions of phase lag angles and local frequencies, and can be extended to systems with time delayed interactions.

1. Introduction

Synchronization is a widespread phenomenon which occurs in systems of networked oscillators in a variety of contexts including physics, biology, ecology and sociology, see for example Arenas, Díaz-Guilera, Kurths, Moreno, and Zhou (2008) and the recent survey (Dörfler & Bullo, 2013). Of interest for these applications are the conditions under which synchronization occurs, such as the allowed parameter sets and the initial values of the system, the means by which synchronization can be controlled, whether the common frequency can be varied, and any factors which prevent the onset of synchronization.

The Kuramoto model (Kuramoto, 1975) has been widely studied as a model of phase synchronization, displaying the main properties of synchronization phenomena while also remaining mathematically tractable. The model consists of a population of weakly-coupled, nearly identical, interacting limit-cycle oscillators which, for sufficiently large coupling, freeze into synchrony. For a detailed discussion see Acebrón, Bonilla, Pérez-Vicente, Ritort, and Spigler (2005) and Arenas et al. (2008), also the introduction in Schmidt, Papachristodoulou, Münz, and Allgöwer (2012) and references therein.

We investigate here the effects of phase lag in the Kuramoto and related models, specifically with respect to the onset of synchronization, the dependence of the frequency on phase lag and, more generally, the properties of phase lag as a control parameter. We analyse in particular the synchronization manifold, comprising solutions of the form (4) (defined as in Arenas et al., 2008, Section 4.1), whereas properties of the time-dependent system such as stability are investigated numerically. The analysis applies to networks with all-to-all coupling with an arbitrary but finite number of nodes N, and is valid for any distribution of natural frequencies and any distribution of phase lag angles. The restriction to finite N is consistent with the approaches taken in Aeyels and Rogge (2004), Chopra and Spong (2009) and Schmidt et al. (2012), rather than the idealized $N \to \infty$ limit used in De Smet and Aeyels (2008), Komarov and Pikovsky (2011), Montbrió and Pazó (2011b), Omel'chenko and Wolfrum (2012), Sakaguchi and Kuramoto (1986), and Wiesenfeld, Colet, and Strogatz (1998), for example, often with a Lorentzian distribution of natural frequencies (Komarov & Pikovsky, 2011; Montbrió & Pazó, 2011b; Pazó & Montbrió, 2011) although, as pointed out in Lafuerza, Colet, and Toral (2010), this can lead to features which are not generic. Our analysis applies also to physical systems for which N is relatively small, as in Nixon et al. (2011) and Nixon et al. (2012). A general aim of this paper, therefore, is to develop methods of analysis which apply for any finite N.

Phase lag parameters occur naturally in complex network models, having been first introduced in the Sakaguchi–Kuramoto (SK) model (Sakaguchi & Kuramoto, 1986) as a means of modelling synchronized systems in which the common frequency differs from the average of the natural frequencies. Phase lag appears in models of Josephson-junction arrays (Wiesenfeld et al., 1998), in

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nonresonant interactions in ensembles of phase oscillators (Komarov & Pikovsky, 2011), in power network systems with nontrivial transfer conductances (Dörfler & Bullo, 2012, see Eq. (2.8)), in the study of mechanical rotors or oscillators (Mertens & Weaver, 2011; Uchida & Golestanian, 2010), in the mean-field version of the complex Ginzburg–Landau equation (Pazó & Montbrió, 2011), in the study of contrarian interactions (Louzada, Araújo, Andrade, & Herrmann, 2012, see Eq. (4)) and in nonlinear quantum networks with interacting qubits (Lohe, 2010, see the Hamiltonian equation (15) where $\alpha + \beta$ is real). Phase lag models are ideal for the study of various synchronization characteristics since the angles can, in particular, behave as control parameters which enable the synchronized system to be tuned to any preset frequency, or else be distributed in such a way as to prevent synchronization occurring at all. Phase lag is equivalent in synchronized systems to time delayed couplings, which occur in biological and other systems, see for example Ares, Morelli, Jörg, Oates, and Jülicher (2012) and Nixon et al. (2012), and so the properties we discuss are relevant to systems with uniform or distributed time delayed interactions. Time delay can also act as a control parameter in order to adjust the frequency of the synchronized system.

We analyse generalizations of the SK model for both uniform and distributed phase lag, and also a model incorporating nonisochronous oscillations with phase lag (Montbrió & Pazó, 2011b). For uniform angles α we obtain an exact frequency formula (Eq. (10)) which shows that the frequency Ω of the phase-locked synchronized system can be adjusted to any value, within preset limits, by varying α, regardless of the fixed natural frequencies ω_i. This property is maintained for distributed phase lag. For the model with nonisochronous oscillations, by contrast, phase lag damps out highly oscillatory nodes, so that the system synchronizes to a frequency which is almost independent of the phase lag.

In Section 2 we discuss the range of models under consideration, including the SK and nonisochronous models with distributed coupling constants of variable sign, and time-delayed interactions. In Section 3 we discuss the SK model in detail, in particular we analyse the synchronization manifold, proving that solutions always exist for sufficiently large couplings κ and arbitrary distributions of frequencies ω_i, for all α with $\cos\alpha > 0$, with some mild assumptions. We describe the numerical procedures used to deduce time-dependent properties of the system, such as the stability of the synchronized solutions, and also to verify the analytic results. We discuss the extension of these results to time-delayed systems in Section 3.5, and also to nonisochronous oscillations with uniform phase lag in Section 3.6. In Section 4 we consider the general case of distributed phase lag angles α_i, finding a simple condition which must be satisfied for synchronization to occur. Section 5 contains brief concluding remarks.

2. Generalized Kuramoto systems with phase lag

We consider a system of N oscillators with variables $\theta_i(t)$ defined by the N equations ($i \in \mathbb{N}_N := \{1, \ldots, N\}$):

$$\dot{\theta}_i = \nu_i + \frac{\kappa_i}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i - \alpha_i), \tag{1}$$

where α_i are phase lag angles and ν_i, the local frequency of the ith oscillator, can depend on κ_i and α_i. The couplings κ_i appear in various models and applications (Dörfler & Bullo, 2012; Lin, Francis, & Maggiore, 2007; Simpson-Porco, Dörfler, & Bullo, 2013), and are usually positive but can also take negative values as discussed in Hong and Strogatz (2011). The sum over j in (1) includes the term $j = i$, which in effect adds $-\kappa_i \sin\alpha_i/N$ to ν_i, although this additional term is small for large N.

One could also include time delayed couplings as in Ares et al. (2012), Schmidt et al. (2012) and Yeung and Strogatz (1999) for example, where the interactions occur through the difference $\theta_j(t - \tau_j) - \theta_i(t)$, where τ_j is a distributed time delay. For synchronized solutions, however, this is equivalent to phase lag in the sense that for solutions of the form (4) we have

$$\theta_j(t-\tau_j) - \theta_i(t) = \phi_j(t) - \phi_i(t) - \alpha_i, \tag{2}$$

where $\alpha_j = \Omega\tau_j$ and $\phi_i(t) = \theta_i(t) - \alpha_i$, i.e. a time delay τ_j has the same effect in the synchronized system as a phase lag α_j, for fixed Ω.

2.1. Two specific models

We consider two special cases of (1), firstly, the SK model (Sakaguchi & Kuramoto, 1986) for which $\kappa_i = \kappa$ is uniform across the network, and the natural frequencies ν_i are independent of κ; secondly, a model with nonisochronicity (Montbrió & Pazó, 2011a,b; Pazó & Montbrió, 2011), derived from a mean-field version of the complex Ginzburg–Landau equation, for which

$$\nu_i \longrightarrow \nu_i + \kappa\tan\alpha_i, \qquad \kappa_i = \frac{\kappa}{\cos\alpha_i}, \tag{3}$$

where $\cos\alpha_i \neq 0$. This model is invariant under $\alpha_i \to \alpha_i + \pi$ and so, without loss of generality and unlike the SK model, we may assume that $|\alpha_i| < \pi/2$ for all i. If the phase lag is uniform, then properties of this model follow from those of the SK model, as outlined in Section 3.6. In both cases κ is a scale parameter and so can be regarded conveniently as a measure of the spread of the frequencies ν_i. We therefore choose ν_i to be of order unity and allow κ to take any positive value. Although previous work has been restricted to the large N limit, as referenced above, significant results have been derived for finite N systems (De Smet & Aeyels, 2007) for partially synchronized systems, and also in Schmidt et al. (2012) where time-delayed models are investigated for finite N.

2.2. Phase-locked synchronization

We investigate phase-locked synchronization in which all oscillators are locked to a common frequency Ω, referred to as *frequency synchronization* in Schmidt et al. (2012, Definition 3.1), which implies that there exists a frequency Ω and angles θ_i^0 such that $\lim_{t\to\infty}\theta_i(t) - \Omega t = \theta_i^0, \forall i \in \mathbb{N}_N$. We therefore look for parameter values for which solutions to (1) exist in the form

$$\theta_i(t) = \Omega t + \theta_i^0. \tag{4}$$

For any set of functions $\theta_i(t)$ define the order parameter $r(t)$ by

$$r(t) := \frac{1}{N}\left|\sum_{i=1}^{N} e^{i\theta_i(t)}\right|. \tag{5}$$

For the solutions (4), attained asymptotically, $r(t)$ is constant and so we define $r_\infty = \lim_{t\to\infty} r(t)$, provided the limit exists. In this case there is an angle ψ^0 such that

$$r_\infty e^{i\psi^0} := \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i^0}, \tag{6}$$

with $0 \leqslant r_\infty \leqslant 1$. The well-known parameter r_∞ was introduced in Kuramoto (1984) and has properties which are discussed in Dörfler and Bullo (2013), see Section 3, particularly 3.3; it measures the phase coherence of the synchronized configuration, for example if r_∞ is close to unity then the phases θ_i^0 are approximately co-located, i.e. the oscillators move in a "single tight clump", but if the oscillators are scattered around the unit circle, then r_∞ is approximately zero, see Strogatz (2000, Section 3.2).

Lemma 2.1. *r_∞ and Ω are determined (explicitly or implicitly) as functions of κ_i by the complex equation*

$$r_\infty = \frac{1}{N}\sum_{i=1}^{N}\exp\left[i\alpha_i + i\sin^{-1}\left(\frac{\Omega - \nu_i}{\kappa_i r_\infty}\right)\right]. \tag{7}$$