Automatica 54 (2015) 284–291

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form ☆

Hui Wang, Quanxin Zhu [1]

School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Nanjing, 210023, PR China

ARTICLE INFO

Article history:
Received 17 March 2014
Received in revised form
2 December 2014
Accepted 30 January 2015
Available online 26 February 2015

Keywords:
Stochastic nonlinear system
Finite-time stability
Backstepping design method
Strict-feedback
High-order system

ABSTRACT

This paper deals with the problem of finite-time stabilization for a class of high-order stochastic nonlinear systems in strict-feedback form. By using Itô's formula, mathematical induction and backstepping design method, a novel state-feedback controller is constructed to guarantee that the closed-loop high-order nonlinear system has a unique solution and the solution of the closed-loop high-order nonlinear system is finite-time stable. A systematic design algorithm is developed for the construction of the backstepping controller. Finally, the effectiveness of the state-feedback controller is illustrated by two examples.

1. Introduction

It is known that the theory of stochastic differential equations has been an important tool in engineering system modeling, analysis and design. Stochastic stability plays an essential role in the theory of stochastic differential equations. A large number of results have been focused on the stability analysis of stochastic differential equations in the literature; see, e.g., Khasminski (1980), Mao (1997) and the references therein.

Recently, a class of stochastic nonlinear systems in strict-feedback form has been paid much attention since many physical devices such as the cart-pendulum system (Mazenc & Bowong, 2003) and the ball–beam with a friction term (Sepulchre, Janković, & Kokotović, 1997) can be described by such a system with strict-feedback structure. By employing the quartic Lyapunov function, Deng and Krstić (1997) firstly discussed the problem of globally asymptotic stabilization for a class of stochastic nonlinear systems in strict-feedback form. Since then, many results on this kind of stochastic nonlinear systems have appeared in the literature; for instance, see, Bresch-Pietri and Krstić (2010), Isidori (1999), Liu and Xie (2013), Marconi and Isidori (2000) and the references therein.

It is obvious that the traditional stability criteria such as stability in probability, moment stability and almost surely stability only can describe the asymptotic behavior of the trajectories of a stochastic system as time goes to infinity. However, these stability criteria fail when we want to know the behavior of the solution of a stochastic system in finite time. As a consequence, finite-time stability for stochastic systems has become popular in recent years (Chen & Jiao, 2010; Yang, Li, & Chen, 2009; Yin, Khoo, Man, & Yu, 2011). By using the state partition of continuous parts of systems, Yang et al. (2009) designed a feedback controller to ensure that a nonlinear stochastic hybrid system is finite-time stochastically stable. Yin et al. (2011) introduced a new definition of finite-time stability for stochastic nonlinear systems and presented a stochastic finite-time stability theorem.

Motivated by the above works, Khoo, Yin, Man, and Yu (2013) studied finite-time stabilization of the following first-order stochastic nonlinear system in strict-feedback form:

$$
\begin{aligned}
dx_1 &= x_2 dt + g_1^T(x_1) dw,\\
dx_2 &= x_3 dt + g_2^T(\bar{x}_2) dw,\\
&\vdots\\
dx_n &= (f(\bar{x}_n) + u) dt + g_n^T(\bar{x}_n) dw,
\end{aligned}
$$

☆ This work was jointly supported by the National Natural Science Foundation of China (61374080), the Natural Science Foundation of Zhejiang Province (LY12F03010), the Natural Science Foundation of Ningbo (2012A610032), and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Changyun Wen under the direction of Editor Miroslav Krstic.

E-mail addresses: wang_8063085@163.com (H. Wang), zqx22@126.com (Q. Zhu).

[1] Tel.: +86 15295591103; fax: +86 2585898785.

http://dx.doi.org/10.1016/j.automatica.2015.02.016
0005-1098/

where $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{F}_t$ is a filtration of sub-σ-fields of $\mathcal{F}$, $\bar{x}_n := (x_1, \dots, x_n)^T = \{\bar{x}_n(t), \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous, adapted $\mathbb{R}^n$-valued measurable process, $w = \{w_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a d-dimensional Brownian motion. Let $\bar{x}_i = (x_1, \dots, x_i)^T$, the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i(\bar{x}_i) : \mathbb{R}^i \to \mathbb{R}^d$, $i = 1, \dots, n$, also called the coefficients of the equation, are assumed to be Borel measurable, continuous, and satisfy $f(0) = 0, g(0) = 0$ for all $t \ge 0, u \in \mathbb{R}$ represents the control input of the system. By using the backstepping design method, Khoo et al. (2013) obtained some sufficient conditions to ensure the finite-time stability of the above first-order stochastic nonlinear system in strict-feedback form. However, finite-time stability of a high-order stochastic nonlinear system in strict-feedback form was not considered in Khoo et al. (2013). Naturally, an important and unsolved problem arose: *Can the backstepping design method on the finite-time stability analysis of stochastic nonlinear systems in strict-feedback form be extended to high-order case?*

In this paper, we will investigate this problem. Different from the first-order stochastic nonlinear system discussed in Khoo et al. (2013), we are concerned with the following high-order stochastic nonlinear system in strict-feedback form:

$$dx_1 = (x_2^{p_1} + f_1(x_1, u, t))dt + g_1^T(x_1)dw,$$
$$dx_2 = (x_3^{p_2} + f_2(\bar{x}_2, u, t))dt + g_2^T(\bar{x}_2)dw,$$
$$\vdots$$
$$dx_n = (f_n(\bar{x}_n, u, t) + u)dt + g_n^T(\bar{x}_n)dw, \tag{1}$$

where p_i, $i = 1, \dots, n-1$, are given odd positive integers, $u \in \mathbb{R}$ is the input of the system, and $f_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, are continuous uncertain functions such that $f_i(0, 0, t) = 0$, $\forall t$.

By constructing a novel Lyapunov function and using Itô's formula, mathematical induction and backstepping design method, a new state-feedback controller is constructed to ensure that the closed-loop high-order nonlinear system has a unique solution and the solution of the closed-loop high-order nonlinear system is finite-time stable. A systematic design algorithm is developed for the construction of the backstepping controller. Finally, two examples are given to show the effectiveness of the theoretical results.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we develop a novel systematic design algorithm and the backstepping design method to achieve finite-time stability of high-order stochastic nonlinear system in strict-feedback form. In Section 4, we use two examples to illustrate the effectiveness of the theoretical results. Finally, in Section 5, we conclude the paper with some general remarks.

2. Preliminaries

In this section, we will introduce some notations and preliminaries. Throughout this paper, unless otherwise specified, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}^n$ denotes the real n-dimensional space, C^2 denotes the continuous functions with continuous derivatives up to second-order, A^T denotes its transpose when A is a given vector or matrix, $\mathrm{tr}\{A\}$ denotes its trace when A is a square matrix, and $|A|$ is the Euclidean norm of a vector A.

Consider the following n-dimensional stochastic nonlinear system:

$$dx = f(x)dt + g^T(x)dw,$$
$$x_0 = \xi, \tag{2}$$

where $f(x)$ and $g(x)$ are continuous in x and satisfy $f(0) = 0$, $g(0) = 0$ for all $t \ge 0$.

Definition 1 (Yin et al., 2011)**.** The trivial solution of (2) is said to be finite-time stable in probability, if the solution exists for any initial data $x_0 \in \mathbb{R}^n$, which is denoted by $x(t, x_0)$, and the following statements hold:

(i) Finite-time attractiveness in probability: for every initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$, the first hitting time $\tau_{x_0} = \inf\{t \mid x(t, x_0) = 0\}$, which is called the stochastic settling time, is finite almost surely, that is, $P\{\tau_{x_0} < \infty\} = 1$;
(ii) Stability in probability: For every pair of $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r) > 0$ such that $P\{|x(t; x_0)| < r$, *for all* $t \ge 0\} \ge 1 - \epsilon$, whenever $|x_0| < \delta$;
(iii) The solution $x(t + \tau_{x_0}; x_0)$ is unique for $t \ge 0$.

Definition 2. A function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing and $\mu(0) = 0$. A class $\mathcal{K}$ function μ is said to belong to class $\mathcal{K}_\infty$ if $\mu(r) \to \infty$ as $r \to \infty$.

For a C^2 Lyapunov function V, let $\mathcal{L}V$ denote the differential operator of V with respect to (2) defined by

$$\mathcal{L}V(x) = \frac{\partial V(x)}{\partial x^T} f(x) + \frac{1}{2}\mathrm{tr}\left\{ g^T(x) \frac{\partial^2 V(x)}{\partial x^T \partial x} g(x) \right\}. \tag{3}$$

Lemma 1 (Khalil, 2002)**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive definite function. Then, there exist two class $\mathcal{K}$ functions μ_1 and μ_2 defined on $[0, +\infty)$ such that*

$$\mu_1(|x|) \le V(x) \le \mu_2(|x|),$$

for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is radially unbounded, then both μ_1 and μ_2 can be chosen to class $\mathcal{K}_\infty$.

Lemma 2 (Yin et al., 2011 and Khoo et al., 2013)**.** *For system (2), if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$, $\mathcal{K}_\infty$ class functions μ_1 and μ_2, positive real numbers $c > 0$ and $0 < \gamma < 1$, such that for all $x \in \mathbb{R}^n$ and $t \ge 0$,*

$$\mu_1(|x|) \le V(x) \le \mu_2(|x|), \tag{4}$$
$$\mathcal{L}V(x) \le -c \cdot (V(x))^\gamma, \tag{5}$$

then the trivial solution of (2) is finite-time attractive and stable in probability.

Lemma 3 (Khoo et al., 2013 and Skorokhod, 1965)**.** *Suppose that $f(x)$ and $g(x)$ are continuous with respect to their variables and satisfy the linear growth condition:*

$$(f(x))^2 + (g(x))^2 \le K(1 + x^2), \tag{6}$$

for $K > 0$. Then for any given $x(t_0)$ independent of $w(t)$, system (2) has a continuous solution with probability 1.

Lemma 4 (Huang, Lin, & Yang, 2005, Li, Jing, & Zhang, 2011 and Lin & Qian, 2002)**.** *Let $x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}^r$, and $p \ge 1$ be a constant. Given any positive real numbers c, d and any real-valued functions $\zeta(x, y) > 0$, $\mu(x, y, z) \ge 0$, the following holds:*

$$\left| x^{\frac{1}{p}} - y^{\frac{1}{p}} \right| \le 2^{\frac{p-1}{p}} |x - y|^{\frac{1}{p}},$$
$$|x \pm y|^p \le 2^{p-1} |x^p \pm y^p|,$$
$$(|x| + |y|)^{\frac{1}{p}} \le |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \le 2^{\frac{p-1}{p}} (|x| + |y|)^{\frac{1}{p}},$$
$$\mu(x, y, z)|x|^c|y|^d \le \frac{c}{c+d}\zeta(x, y)|x|^{c+d} + \frac{d}{c+d}(\mu(x, y, z))^{\frac{c+d}{d}}(\zeta(x, y))^{-\frac{c}{d}}|y|^{c+d}.$$