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The Smooth Decomposition as a nonlinear modal analysis tool

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ABSTRACT

The Smooth Decomposition (SD) is a statistical analysis technique for finding structures in an ensemble of spatially distributed data such that the vector directions not only keep the maximum possible variance but also the motions, along the vector directions, are as smooth in time as possible. In this paper, the notion of the dual smooth modes is introduced and used in the framework of oblique projection to expand a random response of a system. The dual modes define a tool that transforms the SD in an efficient modal analysis tool. The main properties of the SD are discussed and some new optimality properties of the expansion are deduced. The parameters of the SD give access to modal parameters of a linear system (mode shapes, resonance frequencies and modal energy participations). In case of nonlinear systems, a richer picture of the evolution of the modes versus energy can be obtained analyzing the responses under several excitation levels. This novel analysis of a nonlinear system is illustrated by an example.

1. Introduction

The Karhunen–Loève Decomposition (KLD) method, also named Proper Orthogonal Decomposition (POD), has been extensively used as a tool for analyzing random fields. The KLD is a statistical analysis technique for finding the coherent structures in an ensemble of spatially distributed data which defines an optimum basis in terms of energy. It has been advantageously used in different domains as, for example, the stochastic finite elements method [1,2], the simulation of random fields [3], the modal analysis of linear and nonlinear systems [4,5], and construction of reduced-order models [6,7].

A modified decomposition, that is not orthogonal in the Euclidean sense, named Smooth Decomposition (SD), is considered here. The SD can be viewed as a projection of an ensemble of spatially distributed data such that the vector directions of the projection not only keep the maximum possible variance but also the motions resulting along these vector directions are as smooth in time as possible. These vector directions (or structures, or smooth modes) are defined as the eigenvectors of the eigenproblem defined from the correlation matrices of the random field and of the associated time derivative.

The basic idea of this decomposition derives from the optimal tracking approach proposed in [8]. SD was formulated as a multivariate data analysis in [9] and used as a modal analysis tool. Modal analysis of randomly excited system was considered in [10]. The SD approach was developed in cases of time-continuous stationary random vector processes in [11],

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in case of time-continuous stationary random fields in [12] and extended to the time-continuous non-stationary random vector processes in [13].

Recently, SD was also considered to generate reduced bases for discrete nonlinear dynamic systems (see [14,15]) and used in [16] to extract the modal parameters of a vehicle suspension system. The use of data to extract modal parameters was considered in [17].

In this paper, the main properties of the SD are discussed. The new notion of the dual smooth modes is introduced and used in the framework of oblique projection to expand the random response given in the dual smooth expansion. Some optimality properties of this expansion are shown. The parameters of the SD (dual smooth modes and the smooth values) are interpreted in terms of normal modes and resonance frequencies resulting in the modal analysis of a linear system using output-only data. Also, the introduction of the dual modes gives a new picture of the dynamics in terms of energy of the modes. The two pictures, in terms of frequencies and of energies, allows a richer interpretation of the dynamics. This approach overcomes some limitations of the POD. A novel modal analysis of nonlinear system is also proposed.

2. Smooth Decomposition

2.1. Decomposition principle

Let $\{\mathbf{U}(t), t \in \mathbb{R}\}$ be a $\mathbb{R}^n-$valued random process indexed by $\mathbb{R}$. We assume $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ to be a zero-mean second-order stationary ergodic process that admits a time derivative process $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$ which is also a second-order stationary ergodic process. We take $\mathbf{R}_{\mathbf{U}}(\tau) = \mathbb{E}(\mathbf{U}(t+\tau)^T\mathbf{U}(t))$ and $\mathbf{R}_{\dot{\mathbf{U}}}(\tau) = \mathbb{E}(\dot{\mathbf{U}}(t+\tau)^T\dot{\mathbf{U}}(t))$ to denote the covariance matrix function of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$ respectively. We assume that the covariance matrices (of $\mathbf{U}(t)$ and $\dot{\mathbf{U}}(t)$) $\mathbf{R}_{\mathbf{U}}(0)$ and $\mathbf{R}_{\dot{\mathbf{U}}}(0)$ are symmetric positive-definite matrices.

As described in [13], the SD of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ is designed to obtain the most characteristic deterministic vectors $\Gamma(\in \mathbb{R}^n)$ maximizing the ratio between the ensemble average of the inner product between $\mathbf{U}(t)$ and Γ and to the inner product between $\dot{\mathbf{U}}(t)$ and Γ

$$\max_{\mathbf{\Gamma} \in \mathbb{R}^n} J(\mathbf{\Gamma}) \quad \text{with } J(\mathbf{\Gamma}) = \frac{\mathbb{E}((\mathbf{U}(t)^T\mathbf{\Gamma})^2)}{\mathbb{E}((\dot{\mathbf{U}}(t)^T\mathbf{\Gamma})^2)} = \frac{\mathbf{\Gamma}^T\mathbf{R}_{\mathbf{U}}(0)\mathbf{\Gamma}}{\mathbf{\Gamma}^T\mathbf{R}_{\dot{\mathbf{U}}}(0)\mathbf{\Gamma}}. \tag{1}$$

This maximization problem is equivalent to the conditional extreme value problem

$$\max_{\mathbf{\Gamma} \in \mathbb{R}^n} \mathbf{\Gamma}^T\mathbf{R}_{\mathbf{U}}(0)\mathbf{\Gamma} \quad \text{subject to } \mathbf{\Gamma}^T\mathbf{R}_{\dot{\mathbf{U}}}(0)\mathbf{\Gamma} = 1. \tag{2}$$

The objective function $J(\mathbf{\Gamma})$ significantly differs from that used to define the KLD (see for example [18]). Here the denominator of the objective function takes the covariance matrix of the time-derivative process into account. The numerator and the denominator which seem to be time-independent can be related to the time evolution of trajectories of the random processes $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$ over a long time following ergodic property as

$$\mathbb{E}\left((\mathbf{U}(t)^T\mathbf{\Gamma})^2\right) = \lim_{T->\infty}\frac{1}{2T}\int_{-T}^{T}(\mathbf{U}(s)^T\mathbf{\Gamma})^2\, ds \tag{3}$$

$$\mathbb{E}\left((\dot{\mathbf{U}}(t)^T\mathbf{\Gamma})^2\right) = \lim_{T->\infty}\frac{1}{2T}\int_{-T}^{T}(\dot{\mathbf{U}}(s)^T\mathbf{\Gamma})^2\, ds \tag{4}$$

where $\mathbf{U}(s)$ and $\dot{\mathbf{U}}(s)$ in the right-hand-side of the equations denote one of the trajectories and not a random variable as in the left-hand-side (for $\mathbf{U}(t)$ and $\dot{\mathbf{U}}(t)$). From the ergodic point of view, maximizing $J(\mathbf{\Gamma})$ corresponds to find a structure $\mathbf{\Gamma}$ which captures the maximum possible variance in terms of time average of the time displacement field, simultaneously with the minimum possible variance of the time velocity field (in accordance with the drift tracking algorithm proposed in [8]).

The condition for local optimality is given by the gradient of the objective function $J(\mathbf{\Gamma})$ or by the Lagrange multipliers method applied to (2) and reduces to the following generalized eigenproblem:

$$\mathbf{R}_{\mathbf{U}}(0)\mathbf{\Gamma}_k = \lambda_k\mathbf{R}_{\dot{\mathbf{U}}}(0)\mathbf{\Gamma}_k. \tag{5}$$

Due to the properties of the covariance matrices (which are symmetric positive-definite matrices), (5) admits n real positive eigenvalues (λ_k) and the associated eigenvectors satisfy the following properties:

$$\mathbf{\Gamma}_k^T\mathbf{R}_{\mathbf{U}}(0)\mathbf{\Gamma}_l = \mathbf{\Gamma}_k^T\mathbf{R}_{\dot{\mathbf{U}}}(0)\mathbf{\Gamma}_l = 0 \quad \text{for } k \neq l \text{ and } \mathbf{\Gamma}_k^T\mathbf{R}_{\mathbf{U}}(0)\mathbf{\Gamma}_k = \lambda_k\mathbf{\Gamma}_k^T\mathbf{R}_{\dot{\mathbf{U}}}(0)\mathbf{\Gamma}_k.$$

In the sequel we will assume that the eigenvalues λ_k are sorted in descending order and the eigenvectors are scaled to satisfy the constraint condition

$$\mathbf{\Gamma}^T\mathbf{R}_{\dot{\mathbf{U}}}(0)\mathbf{\Gamma} = \boldsymbol{I}_n, \tag{6}$$

where $\mathbf{\Gamma} = [\mathbf{\Gamma}_1\mathbf{\Gamma}_2\ldots\mathbf{\Gamma}_n]$ and $\boldsymbol{I}_n$ denotes the identity matrix.