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Selective pattern formation control: Spatial spectrum consensus and Turing instability approach[☆]

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ABSTRACT

Autonomous pattern formation phenomena are ubiquitous throughout nature. The goal of this paper is to show the possibility to effectively generate various desired spatial patterns by *guiding* such phenomena suitably. To this end, we employ a reaction–diffusion system as a mathematical model, and formulate and solve a novel pattern formation control problem. First, we describe the control objective in terms of spatial spectrum consensus, which enables utilize recent advances on networked control system theory. Next, the effectiveness of the proposed control law is evaluated theoretically by exploiting the center manifold theorem, and also numerically by simulation. The Turing instabilities play a crucial role throughout the paper.

1. Introduction

Regulation of systems to a desired state is a typical control objective. In order to achieve this, negative (often high gain) feedback of the regulation error is widely accepted design policy. On the other hand, there exist a large variety of autonomous pattern formation phenomena in nature. Then, it might be yet another design policy to purposefully guide these phenomena to generate a desired pattern effectively. Actually, it would be a significant contribution to many practical applications, e.g., chemical reactions and systems biology (Camazine, Ristine, Didion, & Thies, 2003; Mikhailov & Showalter, 2006; Murray, 2003), if both the usefulness and also the limitations of such a feedback scheme are revealed. The contribution of this paper is the formulation and solution of a novel pattern formation problem in reaction–diffusion (RD) systems from this point of view.

In order to balance exposition simplicity and intuitive understanding of the resulting phenomena, we investigate an activator–inhibitor RD system with cubic nonlinearities defined on a square domain; see Section 2.1 for its formal definition. This model is simple, however, captures the essential mathematical structure of a large number of superficially different problem settings. As shown in later sections, this model can autonomously generate multiple stationary non-uniform patterns under certain parameter settings. For example, Fig. 1 depicts a sequence of snapshots of one of the spatio-temporal state variables; see Section 3.1 for details. We can observe that the randomly generated initial pattern finally converges to a *roll (stripe) pattern*.

We next present a deeper discussion of the example in Fig. 1, in order to provide more precise insight into our contribution. In Fig. 1 (g)–(j) we can see specific other patterns. While these are transient, their dynamics are slow enough to render them observable. This suggests that we can generate wide variety of spatial patterns by guiding the system's inherent pattern formation mechanism through feedback control, which is the main interest of this paper. Actually, this is shown to be possible in later sections; see Figs. 12 and 14 in Section 4.2 for the stably generated alternative patterns achieved by using Theorems 1–3, the main results of this paper.

It should be emphasized that what we attempt is not to generate arbitrary, given spatial profiles. Instead, we investigate a situation like "generate a hexagonal pattern with an intensity *which fits*

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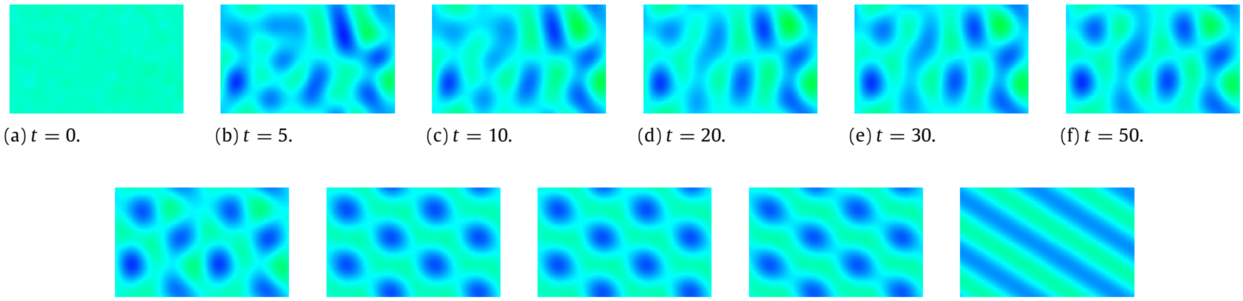

(a) $t = 0$. (b) $t = 5$. (c) $t = 10$. (d) $t = 20$. (e) $t = 30$. (f) $t = 50$.

(g) $t = 100$. (h) $t = 500$. (i) $t = 1000$. (j) $t = 1500$. (k) $t = 3000$.

Fig. 1. Autonomous spatial pattern formation: Normalized snapshots of $\boldsymbol{u}(t, x, y)$.

naturally into the system's inherent pattern formation mechanism." Since the desired pattern includes ones that are hidden, i.e., not observable as a *stationary* pattern (recall Fig. 1(g)–(j)), its exact profile is not available, from which the main drawback of this problem arises. Our approach to circumvent this is summarized as follows:

- In control theoretic terminology, we formulate this problem as a *spatial spectrum consensus with the following two constraints: 1. convergence of the input to zero, and 2. the instability of the origin*. This formulation is not trivial, and allows a simple linear feedback law as well as the proof of its effect in the finite dimensional approximated system.
- In contrast to the approximated finite dimensional analysis, the original PDE, controlled by the proposed control law, cannot satisfy the requirements exactly. In the paper we clarify when, and in which sense, the requirements are satisfied approximately and we obtain the desired pattern formation. The key building block is the center manifold theorem, which proves that *the Turing pattern of the controlled RD system is close to the one that we attempt to generate*.

Note that, we assume spatially distributed sensing/actuation to focus on the theoretical aspects above. Thus, the difficulty of the problem we pose does not lie in the infinite-dimensionality, which certainly makes boundary control problems more challenging (Curtain & Zwart, 1995; Krstic & Smyshlyaev, 2008).

Related works from the controls literature are summarized next. Arcak et al. gave a less conservative sufficient condition to guarantee the *nonexistence* of non-uniform spatial patterns (Arcak, 2011), as well as a specific Turing instability mechanism from a systems biology viewpoint (Hsia, Holtz, Huang, Arcak, & Maharbiz, 2011). However, detailed analysis of the resulting pattern, and control system design, are not fully explored. In the context of distributed parameter system theory, methods were developed for controlling *semilinear* equations (Henry, 1981), including RD systems, from various aspects. In these works, the control objective is usually the stabilization of the origin (as a spatial profile), which shows a clear contrast to our case where we attempt to *generate non-uniform spatial patterns selectively by making an explicit use of the system's instability*.

From a technical viewpoint, in recent years there has been much research effort on diffusively coupled dynamical systems, due to their connection to consensus protocol of multi-agent systems (Fax & Murray, 2004; Mesbahi & Egerstedt, 2010), coupled oscillators (Shafi, Arcak, Jovanović, & Packard, 2013; Stan & Sepulchre, 2007; Steur, Tyukin, & Nijmeijer, 2009), to list a few. Some of these results actually provide useful mathematical tools to show consensus of identical complex-valued subsystems. However, our problem formulation is conceptually different. A preliminary version of this work was presented in Kashima, Ogawa, and Sakurai (2013).

The organization of this paper is as follows. In Section 2 a brief introduction of Turing instabilities is followed by the novel formulation of the feedback control problem for selective pattern formation. In Section 3 we reformulate and solve the problem based on spatial spectrum dynamics; the control law is given in the form of partial diffusive coupling and pinning, both in the spatial spectrum domain. In Section 4 we return to the original RD system to evaluate the pattern formation by the proposed feedback control law. Conclusions are made in Section 5.

Notation. The set of real numbers, complex numbers (with negative real part) and integers are $\mathbb{R}$, $\mathbb{C}$ ($\mathbb{C}_-$) and $\mathbb{Z}$. For $z \in \mathbb{C}$, $\mathrm{Re}\, z$ is its real part. For complex matrix A, we denote the transpose by $A^\top$, the Hermitian conjugate (conjugate transpose) by A^*, the set of eigenvalues by $\mathrm{eig}(A)$, and the maximal singular value by $\|A\|$. We say A is Hurwitz if $\mathrm{eig}(A) \subset \mathbb{C}_-$. The matrix Kronecker product is represented by $\otimes$. For complex vector x, $\|x\| := \sqrt{x^* x}$. The column vector of ones, of compatible dimensions, is denoted by $\mathbb{1} := [1, \ldots, 1]^\top$.

A spatial distribution on the square domain $\Omega := [0, L_x] \times [0, L_y]$ is referred to as a *profile* and represented by bold font such as $\boldsymbol{z}(x, y)$, $(x, y) \in \Omega$. In particular, a profile of specific shape (e.g., roll, hexagonal) is called a *pattern*. The trivial uniform profile $\boldsymbol{z}_{\mathrm{eq}}(x, y) := 0$ on Ω. The Hilbert space $L^2(\Omega)$ is the set of square integrable profiles equipped with the inner product

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_{L^2(\Omega)} := \iint_\Omega \boldsymbol{u}_2^*(x, y) \boldsymbol{u}_1(x, y) dx dy.$$

For $m = (m_x, m_y) \in \mathbb{Z}^2$, we define

$$\boldsymbol{p}_m(x, y) := \frac{1}{\sqrt{L_x L_y}} \exp\left\{ 2\pi \mathrm{j} \left(\frac{m_x x}{L_x} + \frac{m_y y}{L_y} \right) \right\} \tag{1}$$

where $\mathrm{j} := \sqrt{-1}$. This family of scalar functions satisfies $\boldsymbol{p}_m = \boldsymbol{p}_{-m}^*$, and constitutes a complete orthonormal system for $L^2(\Omega)$. For a set $\mathcal{M}$, $-\mathcal{M} := \{-m : m \in \mathcal{M}\}$ and $\pm\mathcal{M} := \mathcal{M} \cup (-\mathcal{M})$. We take $\mathbb{Z}^{2+} \subset \mathbb{Z}^2$ such that $\mathbb{Z}^{2+} \cap (-\mathbb{Z}^{2+}) = \emptyset$ and $\mathbb{Z}^2 = (0, 0) \cup (\pm\mathbb{Z}^{2+})$. The Laplacian operator is $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The (partial) derivative with respect to time t is denoted by the dot; e.g., $\dot{\boldsymbol{z}}(t, x, y)$.

2. Problem formulation

2.1. Mathematical model

Throughout this paper, we investigate the following dynamics of real-valued two-dimensional spatio-temporal variable $\boldsymbol{z}(t, x, y) := [\boldsymbol{u}(t, x, y),\ \boldsymbol{v}(t, x, y)]^\top \in \mathbb{R}^2$ defined on the square domain Ω:

$$\begin{cases} \dot{\boldsymbol{u}} = a_{11}\boldsymbol{u} - a_{12}\boldsymbol{v} - \boldsymbol{u}^3 + d_u \Delta \boldsymbol{u} + \boldsymbol{w}, \\ \dot{\boldsymbol{v}} = a_{21}\boldsymbol{u} - a_{22}\boldsymbol{v} + d_v \Delta \boldsymbol{v}, \end{cases} \tag{2}$$

with the standard periodic boundary conditions