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# Inverse spectral problem for a rod with multiple cracks

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ABSTRACT

A problem of identification of multiple cracks in a rod using spectral data is considered. The cracks are simulated by translational springs. A method for determination of the number of springs, their locations and flexibilities is developed. The problem is reduced to the problem which can be solved by Krein's method. The Krein method enables to reconstruct the damage parameters by means of the use of natural frequencies for longitudinal vibration of the rod with free–free and fixed–free end conditions.



## 1. Introduction

Vibration characteristics of mechanical structures play an important role in the nondestructive testing. A number of publications were devoted to the problems of damage reconstruction in one-dimensional structures, such as rods and beams, using spectral data (natural frequencies). To keep the one-dimensionality of the problem for rods and beams the localized damages are simulated usually with massless springs (translational for tension loads and rotational for bending loads). The first, who showed that cracks can be simulated with massless translational and rotational springs were Rice and Levy [1]. Because we consider below the case of longitudinal vibration, let us mention only some publications where the translational springs were used to simulate localized damages in a longitudinally vibrating rod, see review [2] and papers [3–9]. A relation between the double edge crack sizes and the flexibility of the corresponding translational spring was ascertained in Ref. [7]. Several approaches were developed for solving inverse problem using spectral data. Morassi [3] showed that the highest part of the spectrum determines uniquely (within spatial symmetries) the position of a single crack. This result was extended significantly by Colonello and Morassi [4]. It was shown in the paper that the highest part of the spectrum determines uniquely the (unordered) set of lengths of the segments of rod between the cracks, in case of multiple cracks. Similar results in the uniqueness problem of location of point masses in a vibrating string were obtained by Carlson [10]. Biscontin et al. [5] showed that the asymptotic separation of the spectrum of notched rods can be revealed experimentally. The approach based on the spectrum separation has some limitations. Its application requires the knowledge of natural frequencies of high order. Besides, it is possible to determine not the locations of damages but only the distances between them. Finally, there is no a clear algorithm for separating the branches of natural frequencies. Morassi [6] developed a method for identifying of a single crack using the damage-induced shifts in a pair of natural frequencies.

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Dilena and Morassi [8] proposed a method for identification of a small single crack using natural and antiresonant frequencies. Singh [9] proposed a method for identifying several localized damages using natural frequencies corresponding to free–free and fixed–free end conditions. The method is based on numerical solving of nonlinear equations corresponding to eigenvalue problem. In spite of the fact that the method has led to good results in several considered examples, the method is not rigorously justified because the system of equations may have multiple solutions, or the proposed procedure may not converge to a solution. Besides, it is assumed that the number of localized damages is known, but the problem of determination of the number of damages is one of the most difficult. Thus, rigorously justified and effective methods for the identification of multiple localized damages are not developed till now.

On the other hand, in the fifties of the last century have developed several approaches to the rigorous solution of the inverse Sturm–Liouville problem, with which it is possible to reconstruct the variable coefficients of the ordinary differential equation of the second order by means of the spectral data, see Refs. [11–17].

Shifrin and Ruotolo [18] developed a method for calculation of natural frequencies for a beam with multiple cracks. The method was applied to the case of longitudinal vibration of a rod by Ruotolo and Surace [7]. It was noted in Refs. [18,7] that in the case of solving direct problem, the method leads to very compact determinantal equations for the determination of natural frequencies, but the main advantage of the developed approach consists in the reduction of the differential equations initially given on the intervals between the cracks to a differential equation on the whole interval, occupied by the rod or beam. It enables to reduce the inverse problem to a form, which can be solved by the known methods.

The mathematical formulation of the considered problem is given in Section 2. Reduction of the problem to a form appropriate for solving inverse problem by Krein's method is given in Section 3. Brief description of Krein's method is presented in Section 4. Algorithm for solving inverse problem is given in Section 5. Numerical examples are considered in Section 6. Conclusions are presented in Section 7.

## 2. Statement of the problem

Let us consider a rod of length $l$ and cross-section of constant area $A$. We assume that the rod occupies an interval $0<x<l$ and the translational springs, which simulate the localized damages, are located at points $x_1, x_2, \dots, x_n$ such that $0=x_0<x_1<x_2<\cdots<x_n<x_{n+1}=l$. Denote by $u_j(x)$ the amplitudes of longitudinal displacements under time-harmonic vibration on the interval $x_{j-1}<x<x_j$, where $j=1,2,\dots,n+1$. The equation of harmonic longitudinal oscillations has the following form, see Refs. [5,7]:

$$u_j''(x)+\lambda u_j(x)=0, \quad j=1,2,\dots,n+1, \quad x_{j-1}<x<x_j \tag{1}$$

where $\lambda=(\rho\omega^2/E)$, $E$ is the Young's modulus, $\rho$ is the material density, $\omega$ is a circular frequency.

The conjugation conditions at the locations of springs are of the following form, see Refs. [5,7]:

$$u_j'(x_j)=u_{j+1}'(x_j), \quad u_{j+1}(x_j)-u_j(x_j)=\Delta_j=EAc_ju_j'(x_j), \quad j=1,2,\dots,n \tag{2}$$

where $c_j$ is the flexibility of the $j$th translational spring.

We will consider two types of end conditions. The free–free condition has the form

$$u_1'(0)=0, \quad u_{n+1}'(l)=0 \tag{3}$$

The fixed–free end condition has the form

$$u_1(0)=0, \quad u_{n+1}'(l)=0 \tag{4}$$

Let us denote the eigenvalues of the problems (1), (2), (3) (except $\lambda=0$) by $\lambda_1,\lambda_2,\lambda_3,\dots$ and the eigenvalues of the problems (1), (2), (4) by $\mu_1,\mu_2,\mu_3,\dots$. The problem is to reconstruct the number $n$ of the translational springs, their locations $x_j$ and flexibilities $c_j$, $j=1,2,\dots,n$, using the eigenvalues $\lambda_i$ and $\mu_i$, $i=1,2,\dots$.

## 3. Reformulation of the eigenvalue problem

Analogously to the Refs. [18,7] let us introduce a function $u(x)$, defined on the whole interval $0<x<l$

$$u(x)=u_j(x), \quad x_{j-1}<x<x_j, \quad j=1,2,\dots,n+1 \tag{5}$$

It follows from Eq. (1) and conjugation conditions (2), that function $u(x)$ satisfies the following equation in the interval $0<x<l$ (here and below, the equalities are understood in the sense of distributions):

$$u''(x)+\lambda u(x)-\sum_{k=1}^{n}\Delta_k\delta'(x-x_k)=0 \tag{6}$$

where $\delta(x)$ is Dirac's delta function.

It follows from Eqs. (3) to (5) that end free–free and fixed–free conditions have for the function $u(x)$ the form

$$u'(0)=0, \quad u'(l)=0 \tag{7}$$

$$u(0)=0, \quad u'(l)=0 \tag{8}$$