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Brief paper

Systems of coupled generalized Sylvester matrix equations[☆]

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ABSTRACT

This paper studies some systems of coupled generalized Sylvester matrix equations. We present some necessary and sufficient conditions for the solvability to these systems. We give the expressions of the general solutions to the systems when their solvability conditions are satisfied.

1. Introduction

The generalized Sylvester matrix equations have been attracting much attention from both theoretical and practical points of view. Many problems in control theory can be transformed into the generalized Sylvester matrix equations, such as singular system control (Gavin & Bhattacharyya, 1982; Kwon & Youn, 1987; Shahzad, Jones, Kerrigan, & Constantinides, 2011), robust control (Cavinlii & Bhattacharyya, 1983; Chen, Patton, & Zhang, 1996; Park & Rizzoni, 1994; Varga, 2000), neural network (Zhang, Jiang, & Wang, 2002), Luenberger-type observer design (Duan, Liu, & Thompson, 2001; Tsui, 1988), feedback (Syrmos & Lewis, 1993). A great number of papers have presented several methods for solving generalized Sylvester matrix equations (e.g. Castelan & Gomes da Silva, 2005; Dehghan & Hajarian, 2012; Ding & Chen, 2006; El Guennouni, Jbilou, & Riquet, 2002; Hu & Reichel, 1992; Kägström, 1994; Song & Chen, 2011; van der Woude, 1987; Wang, Sun, & Li, 2002; Wang, van der Woude, & Chang, 2009; Wu, Duan, & Zhou, 2008; Xie, Ding, & Ding, 2009; Zhou & Duan, 2008).

The study on the mixed generalized Sylvester matrix equations is active in recent years. Lee and Vu (2012) derived a necessary and sufficient solvability condition for the system

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2, \end{cases} \tag{1}$$

where A_i, B_i and C_i $(i = 1, 2)$ are given complex matrices, X, Y and Z are variable matrices. Liu (2006) gave a solvability condition to (1). Recently, Wang and He (2013) presented a new necessary and sufficient solvability condition for the system (1), and gave an expression of the general solution when it is solvable. The admissible ranks of the solution to (1) were presented by Wang and He (2013). He and Wang (2014) gave some necessary and sufficient solvability conditions for the system

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \end{cases} \tag{2}$$

and derived an expression of the general solution to (2). When $X = Z$, the system (1) becomes pairs of generalized Sylvester equations

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2X - YB_2 = C_2. \end{cases} \tag{3}$$

Wimmer (1994) gave a necessary and sufficient condition for the existence of a simultaneous solution of (3). Kägström (1994) obtained a solution of (3) by using generalized Schur methods.

Motivated by the wide application of generalized Sylvester matrix equation and in order to improve the theoretical development

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of generalized Sylvester matrix equations, we consider the coupled generalized Sylvester matrix equations

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2, \\ A_3Z - WB_3 = C_3, \end{cases} \quad (4)$$

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \\ A_3Z - WB_3 = C_3, \end{cases} \quad (5)$$

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \\ A_3W - ZB_3 = C_3, \end{cases} \quad (6)$$

where A_i, B_i and C_i $(i = 1, 2, 3)$ are given complex matrices, X, Y, Z and W are variable matrices.

The main contribution of this paper is to present some solvability conditions and the expressions of the general solutions to the above-mentioned systems of coupled generalized Sylvester matrix equations (4)–(6).

The paper is organized as follows. In Section 2, we give some known results and lemmas. In Section 3, we consider the solvability conditions and the expressions of the general solutions to the coupled generalized Sylvester matrix equations (4)–(6).

Throughout this paper, we denote the complex number field by $\mathbb{C}$. The notation $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices. The identity matrix with an appropriate size is denoted by I. For a complex matrix A, the symbols A^* and $r(A)$ stand for the conjugate transpose and rank of A, respectively. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X to the following four matrix equations

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX, \qquad (XA)^* = XA.$$

Furthermore, L_A and R_A stand for the two projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by A, respectively.

2. Preliminaries

We first review some lemmas which are used in the further development of this paper. Observe that

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \quad (7)$$

can take a key role in investigating the general solutions to the coupled generalized Sylvester matrix equations (4)–(6).

Lemma 2.1 (Wang & He, 2012). *Let $A_1, B_1, C_3, D_3, C_4, D_4$, and E_1 be given. Set*

$$A = R_{A_1}C_3, \quad B = D_3L_{B_1}, \quad C = R_{A_1}C_4, \quad D = D_4L_{B_1},$$
$$E = R_{A_1}E_1L_{B_1}, \quad M = R_AC, \quad N = DL_B, \quad S = CL_M.$$

Then the following statements are equivalent:

(1) *Eq. (7) is consistent.*

(2)

$$R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

In this case, the general solution can be expressed as

$$X_1 = A_1^\dagger(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^\dagger T_7B_1 + L_{A_1}T_6,$$
$$X_2 = R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^\dagger + A_1A_1^\dagger T_7 + T_8R_{B_1},$$
$$X_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger ST_2R_NDB^\dagger + L_AT_4 + T_5R_B,$$
$$X_4 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

where $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Lemma 2.2 (Baksalary & Kala, 1979). *Let A_1, B_1, and C_1 be given. Then the Sylvester matrix equation*

$$A_1X - YB_1 = C_1 \quad (8)$$

is consistent if and only if $R_{A_1}C_1L_{B_1} = 0$, i.e. $r\begin{bmatrix} C_1 & A_1 \\ B_1 & 0 \end{bmatrix} = r(A_1) + r(B_1)$. In this case, the general solution to (8) can be expressed as

$$X = A_1^\dagger C_1 + U_1B_1 + L_{A_1}U_2,$$
$$Y = -R_{A_1}C_1B_1^\dagger + A_1U_1 + U_3R_{B_1},$$

where U_1, U_2, and U_3 are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Lemma 2.3 (Marsaglia & Styan, 1974). *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{m\times k}, C \in \mathbb{C}^{l\times n}, D \in \mathbb{C}^{m\times p}, E \in \mathbb{C}^{q\times n}, Q \in \mathbb{C}^{m_1\times k}$, and $P \in \mathbb{C}^{l\times n_1}$ be given. Then*

(1) $r(A) + r(R_AB) = r(B) + r(R_BA) = r[A \quad B]$.

(2) $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{bmatrix} A \\ C \end{bmatrix}$.

3. Some solvability conditions and the general solutions to systems (4)–(6)

The purpose of this section is twofold. First, we give some necessary and sufficient conditions for the consistence to systems (4)–(6). Second, we give the expressions of the general solutions to the above-mentioned systems of coupled generalized Sylvester matrix equations.

Theorem 3.1. *Let A_i, B_i, and C_i $(i = 1, 2, 3)$ be given. Set*

$$A_4 = A_2L_{A_3}, \quad B_4 = R_{B_1}B_2, \quad A = R_{A_4}A_2, \quad B = B_3L_{B_4},$$
$$C = R_{A_4}A_1, \quad D = B_2L_{B_4}, \quad M = R_AC, \quad N = DL_B,$$
$$S = CL_M,$$
$$C_4 = C_2 - A_2A_3^\dagger C_3 - R_{A_1}C_1B_1^\dagger B_2, \quad E = R_{A_4}C_4L_{B_4}.$$

Then the following statements are equivalent:

(1) *The system of coupled generalized Sylvester matrix equations (4) is consistent.*

(2)

$$R_{A_i}C_iL_{B_i} = 0, \quad (k = 1, 3), \quad R_MR_AE = 0,$$
$$EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

(3)

$$R_{A_j}C_jL_{B_j} = 0, \quad (j = 1, 2, 3), \quad R_MR_AE = 0,$$
$$EL_BL_N = 0, \quad R_CEL_B = 0.$$

(4)

$$r\begin{bmatrix} C_i & A_i \\ B_i & 0 \end{bmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3),$$

$$r\begin{bmatrix} A_1 & A_2 & C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{bmatrix} = r[A_1 \quad A_2] + r[B_1 \quad B_2], \quad (9)$$

$$r\begin{bmatrix} B_2 & 0 \\ B_3 & 0 \\ C_2 & A_2 \\ C_3 & A_3 \end{bmatrix} = r\begin{bmatrix} A_2 \\ A_3 \end{bmatrix} + r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \quad (10)$$

$$r\begin{bmatrix} C_2 & C_1 & A_1 & A_2 \\ C_3 & 0 & 0 & A_3 \\ B_2 & B_1 & 0 & 0 \\ B_3 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} + r\begin{bmatrix} B_2 & B_1 \\ B_3 & 0 \end{bmatrix}. \quad (11)$$