Automatica 50 (2014) 2872–2879

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On the quadratic stability of switched linear systems associated with symmetric transfer function matrices[☆]

Yashar Kouhi [a,1], Naim Bajcinca [a,b], Jörg Raisch [a,b], Robert Shorten [c]

[a] Technische Universität Berlin, Control Systems Group, Einsteinufer 17, Sekr. EN 11, 10587 Berlin, Germany
[b] Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstr. 1, 39106 Magdeburg, Germany
[c] IBM Research Ireland, Ireland

ARTICLE INFO

Article history:
Received 30 August 2013
Received in revised form
10 February 2014
Accepted 18 June 2014
Available online 30 September 2014

Keywords:
Switched linear systems
Strictly positive real systems
Symmetric transfer function matrices

ABSTRACT

In this paper we give necessary and sufficient conditions for weak and strong quadratic stability of a class of switched linear systems consisting of two subsystems, associated with symmetric transfer function matrices. These conditions can simply be tested by checking the eigenvalues of the product of two subsystem matrices. This result is an extension of the result by Shorten and Narendra for strong quadratic stability, and the result by Shorten et al. on weak quadratic stability for switched linear systems. Examples are given to illustrate the usefulness of our results.

1. Introduction

Consider the switched linear system

$$\Sigma_\sigma : \dot{x} = (1-\sigma(t))A_1x + \sigma(t)A_2x, \quad \sigma(t) \in \{0,1\}, \tag{1}$$

where A_1 and A_2 are constant matrices with real entries, and $\sigma(t)$ is an arbitrary time switching signal which assumes a finite number of switchings within a finite time interval. Let P be a symmetric positive definite matrix satisfying

$$A_i^TP + PA_i = -Q_i, \quad i \in \{1,2\}. \tag{2}$$

Then, the function $V(x) = x^TPx$ is said to be a *strong* common quadratic Lyapunov function (CQLF) for the switched system (1) if the Q_i's are both positive definite. For the purpose of this paper, V is said to be a *weak* common quadratic Lyapunov function if both of the Q_i's are positive semi-definite and exactly one of them is singular. If such a CQLF exists the switched system (1) is called strongly quadratically stable implying that all solutions converge exponentially to zero, or weakly quadratically stable implying that all solutions are bounded (Note that strong quadratic stability is a sufficient condition for exponential stability of switched linear systems. For more results in this respect, see, *e.g.*, Hespanha and Morse (1999); Zhang and Huijun (2010).) In this paper, we wish to determine conditions on A_1 and A_2 such that a CQLF exists. Such stability problems arise in a variety of applications; see, for example, Liberzon (2003); Lin and Antsaklis (2009). Over the past decade many techniques have been developed to study CQLF existence problems. The most notable among these techniques are related to the use of Linear Matrix Inequalities (LMI) in the context of convex optimization. While LMI's and other numerical techniques are useful, usually they offer little insight into when such functions exist, and their extensions to cases where one or more of the system matrices is singular are problematic. Thus it is of interest to develop algebraic conditions that can be used to check for the existence of common quadratic Lyapunov functions. Initial results in this direction were given in Shorten, Corless, Wulff, Klinge, and Middleton (2009) and Shorten and Narendra (2003), where it was shown that any two matrices, one of which is Hurwitz and the other one has all eigenvalues in the open left half plane and exactly one eigenvalue at the origin, and differ by a rank-1 matrix, will admit a weak CQLF provided that the matrix product A_1A_2 has no real negative eigenvalues and exactly one zero eigenvalue. Despite much effort it has not been possible to develop similar results for more general matrix pairs. An alternative approach therefore is to seek pairs of matrices for which similar conditions hold true. In this paper we identify one such class.

☆ The material in this paper was partially presented at the European Control Conference (ECC2013), July 17–19 2013, Zurich, Switzerland. This paper was recommended for publication in revised form by Associate Editor Michael V. Basin under the direction of Editor Ian R. Petersen.

E-mail addresses: kouhi@control.tu-berlin.de (Y. Kouhi), bajcinca@mpi-magdeburg.mpg.de (N. Bajcinca), raisch@control.tu-berlin.de (J. Raisch), robshort@ie.ibm.com (R. Shorten).

[1] Tel.: +49 0 30 314 79281; fax: +49 0 30 314 21137.

http://dx.doi.org/10.1016/j.automatica.2014.09.007
0005-1098/

Specifically, we are interested in systems that arise in the context of symmetric transfer function matrices. Symmetric transfer function matrices are seen frequently, for example in the study of electrical circuits (Helmke, Rosenthal, & Wang, 2006; Semlyen & Gustavsen, 2009). Switched systems arise in situations where intermittent feedback is used to control such plants. The principal contribution in the note is to demonstrate that the Kalman–Yakubovic–Popov lemma is tight for such systems. We then use this observation to recover compact spectral conditions for the existence of a common (strong or weak) quadratic Lyapunov function for a certain class of switched systems. Finally, before proceeding it is important to note that even though this paper follows Shorten and Narendra (2003) and Shorten et al. (2009) in spirit (albeit for a much more general system class), the extension presented here does not immediately follow from these results, and is highly non-trivial involving detailed and original mathematical arguments.

Our paper is structured as follows. We conclude this section with the mathematical notation used in the paper. We then present our problem statement and some preliminary results that we shall use. We then present our main result and give some examples to illustrate their use.

Notation: Throughout, $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively. We denote n-dimensional real Euclidean space by $\mathbb{R}^n$ and the space of $n \times n$ matrices with real entries by $\mathbb{R}^{n\times n}$. We denote a state space representation of the $m \times m$ transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ by (A, B, C, D), where we always assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ has full column rank, $C \in \mathbb{R}^{m\times n}$ has full row rank, and $D \in \mathbb{R}^{m\times m}$ for some $n \geq m$. The inequality $Q \geq 0$ (respectively $Q > 0$) denotes that the matrix Q is positive semi-definite (respectively positive definite). $\otimes$ denotes the Kronecker product of two matrices, and v^* denotes the conjugate transpose of a vector $v \in \mathbb{C}^n$. Let y_{ij} be the (i, j) element of the matrix $Y \in \mathbb{R}^{n\times n}$. Then, we define the vectorization of Y as $\text{vec}(Y) = [y_{11} \dots y_{n1}\, y_{12} \dots y_{n2} \dots y_{1n} \dots y_{nn}]^T$.

2. Problem statement: switched systems associated with symmetric transfer function matrices

We are interested in determining the existence of a CQLF for the following class of switched system: $\dot{x} = A(t)x$ where $A(t)$ is a matrix valued function taking the values A_1 or A_2 which are related through a symmetric transfer function matrix. Moreover, we assume that A_1 is Hurwitz (all eigenvalues have negative real part) and that A_2 has eigenvalues that are either in the open left half plane or at the origin. It is convenient to rewrite these matrices as $A_1 := A$, and $A_2 := A - BD^{-1}C$, and using this notation, to associate a transfer function matrix $G(s) = G^T(s) = C(sI - A)^{-1}B + D$ with the system. Thus, with this choice of A_1 and A_2 the switched system (1) is reformulated as

$$\Sigma_\sigma : \dot{x} = (A - \sigma(t)BD^{-1}C)x, \quad \sigma(t) \in \{0, 1\}. \tag{3}$$

Our interest in this paper is when $G(s)$ is symmetric for two principal reasons.

(i) Symmetric systems: First, symmetric transfer function matrices are ubiquitous in the study of electrical systems (Helmke et al., 2006; Semlyen & Gustavsen, 2009), and in systems with collocated sensors and actuators (Yang & Qiu, 2002). They are also found in the study of chemical process plants (Shinskey, 1984). The study of switched systems is important as symmetry of a transfer function matrix is often preserved under symmetric feedback. For example, consider the state space realization of a symmetric transfer function

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx. \end{cases} \tag{4}$$

Suppose now that intermittent output feedback of the form $u = -\sigma(t)Ky$ with $\sigma(t) \in \{0, 1\}$ and K a symmetric matrix is used to control the plant. Thus the closed loop system is of the form of $\dot{x} = (A - \sigma(t)BKC)\,x$ which is in the form of (3) if K is invertible. Such a scenario may readily occur whenever communication through which feedback is transmitted is unreliable. As is well known the stability of this system is not guaranteed, unless one can show the existence of a Lyapunov function.

(ii) Correspondence classes: A second motivation for the study of this system class comes from the definition of the matrices $A_1 = A$ and $A_2 = A - BD^{-1}C$. Many switched systems may be put in the form of this class by an appropriate choice of matrices B, C, and D. Thus, if we can establish results for the class of switched systems (3), then the same results can be used to determine quadratic stability of a much wider class of switched systems. Note, precisely which class of systems is isomorphic to the class considered in this paper is characterized by the following lemma (the proof is given in the Appendix).

Lemma 1. *Consider two matrices $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{n\times n}$. A sufficient condition for the existence of real matrices A, B, C, and $D = D^T > 0$ which satisfy $A_1 = A$, $A_2 = A - BD^{-1}C$, and $G(s) = G^T(s) = C(sI - A)^{-1}B + D$ is that the two matrices*

$$E_1 := I \otimes A_1 - A_1 \otimes I \quad \text{and}$$
$$E_2 := I \otimes A_2 - A_2 \otimes I \tag{5}$$

share a common eigenvector corresponding to a zero eigenvalue, say $vec(Y) = [y_{11} \dots y_{n1}\ y_{12} \dots y_{n2} \dots y_{1n} \dots y_{nn}]^T$, such that Y is symmetric and invertible, and $(A_1 - A_2)Y$ is positive semi-definite. Furthermore, if (A, B, C, D) is a minimal realization of $G(s)$, then this sufficient condition is also necessary.

3. Definitions and preliminary results

In this section we present several general results and definitions that are useful in proving the principal result of this note.

(i) *Strict positive realness:* An $m \times m$ rational transfer function matrix $G(s)$ is said to be strictly positive real (SPR) if there exists a real scalar $\alpha > 0$ such that $G(s)$ is analytic for $\text{Re}(s) \geq -\alpha$ and

$$G(j\omega - \alpha) + G^T(-j\omega - \alpha) \geq 0, \quad \forall \omega \in \mathbb{R}; \tag{6}$$

see Corless and Shorten (2010) and Zhou, Doyle, and Glover (1996). The following characterization, inspired principally by Narendra and Taylor (1973), provides a more convenient description of a SPR transfer function matrix.

Lemma 2 (*See Corless & Shorten, 2010*)**.** *Suppose A is Hurwitz. Then the $m \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is strictly positive real if and only if*

$$G(j\omega) + G^T(-j\omega) > 0, \quad \omega \in \mathbb{R}, \tag{7}$$

$$\lim_{\omega\to\infty} \omega^{2(m-p)} \det(G(j\omega) + G^T(-j\omega)) > 0, \tag{8}$$

where $p = rank(G(\infty) + G^T(\infty))$.

(ii) *Kalman–Yakubovic–Popov lemma (KYP):* A basic result in systems theory is the KYP lemma. The KYP lemma gives algebraic conditions for the existence of a certain type of Lyapunov functions; see, *e.g.*, Boyd, El Ghaoui, Feron, and Balakrishnan (1994).

Lemma 3 (*KYP Lemma, See Boyd et al., 1994*)**.** *Let A be Hurwitz, (A, B) be controllable, and (A, C) be observable. Then $G(s) = C(sI -$