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Brief paper

Locally optimal controllers and globally inverse optimal controllers☆

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ABSTRACT

In this paper we consider the problem of global asymptotic stabilization with prescribed local behavior. We show that this problem can be formulated in terms of control Lyapunov functions. Moreover, we show that if the local control law has been synthesized employing an LQ approach, then the associated Lyapunov function can be seen as the value function of an optimal problem with some specific local properties. We illustrate these results on two specific classes of systems: backstepping and feedforward systems. Finally, we show how this framework can be employed when considering the orbital transfer problem.

1. Introduction

The synthesis of a stabilizing control law for systems described by nonlinear differential equations has been the subject of great interest by the nonlinear control community during the last three decades. Depending on the structure of the model, some techniques are now available to synthesize control laws ensuring global and asymptotic stabilization of the equilibrium point.

For instance, we can refer to the popular backstepping approach (see Andrieu & Praly, 2008, Krstic, Kanellakopoulos, & Kokotovic, 1995 and references therein), or the forwarding approach (see Jankovic, Sepulchre, & Kokotovic, 1996, Mazenc & Praly, 1996 and Praly, Ortega, & Kaliora, 2002) and some others based on energy considerations or dissipativity properties (see Kokotović & Arcak, 2001 for a survey of the available approaches).

Although the global asymptotic stability of the steady state can be achieved in some specific cases, it remains difficult to address in the same control objective performances issues of a nonlinear system in a closed loop. However, when the first order approximation of the non-linear model is considered, some performances aspects can be addressed by using linear optimal control techniques (using LQ controller for instance).

Hence, it is interesting to raise the question of synthesizing a nonlinear control law which guarantees the global asymptotic stability of the origin while ensuring a prescribed local linear behavior. For instance, this problem has been addressed by Ezal, Pan, and Kokotovic (2000). In this paper local optimal control laws are designed for systems which admit the existence of a backstepping.

In the present paper we consider this problem in a general manner. In a Section 1 we will motivate this control problem and we will consider a first strategy based on the design of a uniting control Lyapunov function. We will show that this is related to an equivalent problem which is the design of a control Lyapunov function with a specific property on the quadratic approximation around the origin. In the second part of this paper, we will consider the case in which the prescribed local behavior is an optimal LQ controller. In this framework, we investigate what type of performances is achieved by the control solution to the stabilization with prescribed local behavior. In the third part we consider two specific classes of systems and show how the control with prescribed local behavior can be solved. With our new context we revisit partially results obtained by Ezal et al. (2000). Finally in the fourth part of the paper, we consider a specific control problem which is the orbital transfer problem. Employing the Lyapunov approach of Kellett and Praly (2004) we exhibit a class of costs for which the stabilization with local optimality can be achieved.

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2. Stabilization with prescribed local behavior

To present the problem under consideration, we introduce a general controlled nonlinear system described by the following ordinary differential equation:

$$\dot{\mathcal{X}} = \Phi(\mathcal{X}, u), \tag{1}$$

with the state $\mathcal{X}$ in $\mathbb{R}^n$ and $\Phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a C^1 function such that $\Phi(0, 0) = 0$ and u in $\mathbb{R}^p$ is a control input. For this system, we can introduce the two matrices $\mathbb{A}$ in $\mathbb{R}^{n\times n}$ and $\mathbb{B}$ in $\mathbb{R}^{n\times p}$ describing its first order approximation: $\mathbb{A} := \frac{\partial \Phi}{\partial \mathcal{X}}(0, 0)$, $\mathbb{B} := \frac{\partial \Phi}{\partial u}(0, 0)$. All along the paper hidden in our assumptions, the couple $(\mathbb{A}, \mathbb{B})$ is assumed to be stabilizable.

For system (1), the problem we intend to solve can be described as follows:

Global asymptotic stabilization with prescribed local behavior: Let a linear state feedback law $u = K_o\mathcal{X}$ with K_o in $\mathbb{R}^{p\times n}$ which stabilizes the first order approximation of system (1) (i.e. $\mathbb{A}+\mathbb{B}K_o$ is Hurwitz) be given. We are looking for a stabilizing control law $u = \alpha_o(\mathcal{X})$, with $\alpha_o : \mathbb{R}^n \to \mathbb{R}^p$, a locally Lipschitz map differentiable at 0 such that:

(1) The origin of the closed-loop system $\dot{\mathcal{X}} = \Phi(\mathcal{X}, \alpha_o(\mathcal{X}))$ is globally and asymptotically stable;
(2) The first order approximation of the control law α_o satisfies the following equality.

$$\frac{\partial \alpha_o}{\partial \mathcal{X}}(0) = K_o. \tag{2}$$

This problem has already been addressed in the literature. For instance, it is the topic of the papers by Benachour, Andrieu, Praly, and Hammouri (2011), Ezal et al. (2000), Sahnoun, Andrieu, and Nadri (2012). Note moreover that this subject can be related to the problem of uniting a local and a global control law as introduced by Teel and Kapoor (1997) (see also Prieur, 2001).

In this paper, we restrict our attention to the particular case in which the system is input affine. More precisely we consider systems in the form

$$\dot{\mathcal{X}} = a(\mathcal{X}) + b(\mathcal{X})u, \tag{3}$$

with the two C^1 functions $a : \mathbb{R}^n \to \mathbb{R}^n$ and $b : \mathbb{R}^n \to \mathbb{R}^{n\times p}$. In this case we get $\mathbb{A} = \frac{\partial a}{\partial \mathcal{X}}(0)$ and $\mathbb{B} = b(0)$.

Employing the tools developed by Andrieu and Prieur (2010) it is possible to show that merging control Lyapunov function may solve the problem of stabilization with prescribed local behavior. In the following, we show that working with the control Lyapunov function is indeed equivalent to address this problem.

Theorem 1. *Given a linear state feedback law $u = K_o\mathcal{X}$ with K_o in $\mathbb{R}^{p\times n}$ which stabilizes the first order approximation of system (3). The following two statements are equivalent.*

(1) *There exists a locally Lipschitz function $\alpha_o : \mathbb{R}^n \to \mathbb{R}^p$ solution to the* global asymptotic stabilization with prescribed local behavior problem.
(2) *There exists a C^2 proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that the following two properties are satisfied.*
  - *If we denote[2] $P := \frac{1}{2}H(V)(0)$, then P is a positive definite matrix. Moreover this inequality holds.*
  $$(\mathbb{A} + \mathbb{B}K_o)'P + P(\mathbb{A} + \mathbb{B}K_o) < 0; \tag{4}$$
  - *Artstein condition is satisfied. More precisely, this implication holds for all $\mathcal{X}$ in $\mathbb{R}^n \setminus \{0\}$,*
  $$L_bV(\mathcal{X}) = 0 \Rightarrow L_aV(\mathcal{X}) < 0, \tag{5}$$
  where $L_bV(\cdot) = \partial V/\partial \mathcal{X} \cdot b(\cdot)$, and L_aV is analogously defined.

Proof. (1) ⇒ (2) The proof of this part of the theorem is based on recent results obtained in Andrieu and Prieur (2010). Indeed, the design of the function V is obtained from the uniting of a quadratic local control Lyapunov function (denoted V_0) and a global control Lyapunov function (denoted V_∞) obtained employing a converse Lyapunov theorem.

First of all, employing the converse Lyapunov theorem of Kurzweil (1956), there exists a C^∞ function $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ such that $\frac{\partial V_\infty}{\partial \mathcal{X}}(\mathcal{X})[a(\mathcal{X}) + b(\mathcal{X})\alpha_o(\mathcal{X})] < 0,\ \forall\ \mathcal{X} \neq 0$. On the other hand, $\mathbb{A} + \mathbb{B}K_o$ being Hurwitz, there exists a matrix P such that the algebraic Lyapunov inequality (4) is satisfied. Let V_0 be the quadratic function $V_0(\mathcal{X}) = \mathcal{X}'P\mathcal{X}$. Due to the fact that K_o satisfies Eq. (2) it yields that the matrix $\mathbb{A} + \mathbb{B}K_o$ is the first order approximation of the system (3) with the control law $u = \alpha_o(x)$. Consequently, it implies that there exists a positive real number ϵ_1 such that $\frac{\partial V_0}{\partial \mathcal{X}}(\mathcal{X})[a(\mathcal{X}) + b(\mathcal{X})\alpha_o(\mathcal{X})] < 0,\ \forall\ 0 < |\mathcal{X}| \leq \epsilon_1$. This implies that the time derivative of the two control Lyapunov functions V_0 and V_∞ can be made negative definite with the same control law in a neighborhood of the origin. Employing Andrieu and Prieur (2010, Theorem 2.1), it yields the existence of a function $V : \mathbb{R}^n \to \mathbb{R}_+$ which is C^2 at the origin and a positive real number ϵ_2 such that the following two properties hold.

- For all $\mathcal{X}$ in $\mathbb{R}^n \setminus \{0\}$, $\frac{\partial V}{\partial \mathcal{X}}(\mathcal{X})[a(\mathcal{X}) + b(\mathcal{X})\alpha_o(\mathcal{X})] < 0$. Hence, Eq. (5) is satisfied;
- For all $\mathcal{X}$ in $\mathbb{R}^n$ such that $|\mathcal{X}| \leq \epsilon_2$, we have $V(\mathcal{X}) = V_0(\mathcal{X})$. Consequently $\mathcal{H}(V)(0) = 2P$.

(2) ⇒ (1) Let Q be the positive definite matrix defined as, $Q := -(\mathbb{A} + \mathbb{B}K_o)'P + P(\mathbb{A} + \mathbb{B}K_o)$. Employing the local approximation of the Lyapunov function V, it is possible to find r_0 such that

$$L_aV(\mathcal{X}) + L_bV(\mathcal{X})K_o\mathcal{X} < 0, \quad \forall \mathcal{X} \in \{0 < V(\mathcal{X}) \leq r_0\}.$$

This implies that the control Lyapunov function V satisfies the small control property (see Sontag, 1989). Hence, we get the existence of a control law α_∞ (given by Sontag's universal formulae introduced by Sontag, 1989) such that this one satisfies for all $\mathcal{X} \neq 0$

$$L_aV(\mathcal{X}) + L_bV(\mathcal{X})\alpha_\infty(\mathcal{X}) < 0.$$

A solution to the stabilization with prescribed local problem can be given by the control law $\alpha_o(\mathcal{X}) = \rho(V(\mathcal{X}))\alpha_\infty(\mathcal{X}) + (1 - \rho(V(\mathcal{X})))K_o\mathcal{X}$ where $\rho : \mathbb{R}_+ \to [0, 1]$ is any locally Lipschitz function such that $\rho(s) = \begin{cases} 0, & s \leq \frac{r_0}{2} \\ 1, & s \geq r_0. \end{cases}$ Note that with this selection, it yields that equality (2) holds. Moreover, we have along the solution of the system (3)

$$\begin{aligned}\dot{V}(\mathcal{X})|_{u=\alpha_o(\mathcal{X})} &= \rho(V(\mathcal{X}))\dot{V}(\mathcal{X})|_{u=\alpha_\infty} \\ &\quad + (1 - \rho(V(\mathcal{X})))\dot{V}(\mathcal{X})|_{u=K_o\mathcal{X}} < 0.\end{aligned}$$

Hence, we get the result. □

From this theorem, we see that looking for a global control Lyapunov function locally assigned by the prescribed local behavior and looking for the controller itself are equivalent problems.

3. Locally optimal and globally inverse optimal control laws

If one wants to guarantee a specific behavior on the closed loop system, one might want to find a control law which minimizes a

[2] In the following, given a C^2 function $V : \mathbb{R}^n \to \mathbb{R}$, the notation $H(V)(\mathcal{X})$ is the Hessian matrix in $\mathbb{R}^{n\times n}$ evaluated at $\mathcal{X}$ of the function V. More precisely, it is the matrix $(H(V))_{i,j}(\mathcal{X}) = \frac{\partial^2 V}{\partial \mathcal{X}_i \partial \mathcal{X}_j}(\mathcal{X})$.