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Computing multiple periodic solutions of nonlinear vibration problems using the harmonic balance method and Groebner bases

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ABSTRACT

This paper is devoted to the study of vibration of mechanical systems with geometric nonlinearities. The harmonic balance method is used to derive systems of polynomial equations whose solutions give the frequency component of the possible steady states. Groebner basis methods are used for computing all solutions of polynomial systems. This approach allows to reduce the complete system to an unique polynomial equation in one variable driving all solutions of the problem. In addition, in order to decrease the number of variables, we propose to first work on the undamped system, and recover solution of the damped system using a continuation on the damping parameter. The search for multiple solutions is illustrated on a simple system, where the influence of the retained number of harmonic is studied. Finally, the procedure is applied on a simple cyclic system and we give a representation of the multiple states versus frequency.

1. Introduction

The present work is devoted to studying nonlinear dynamic systems subjected to polynomial nonlinearities. It is well known that nonlinear systems may exhibit complex dynamics and, in particular, multiple steady-state solutions [1–3]. Some nonlinear systems [1] can even have a countable infinity of periodic solutions, which makes the search for all solutions very difficult, if not impossible. The goal of this paper is to propose a method based on the harmonic balance method (HBM), Groebner bases and continuation methods that allow deriving multiple solutions of a nonlinear dynamic system (free or forced). As the HBM introduce an approximation (truncation in the number of retained harmonics), only a finite number of solution can be obtained, and here, “multiple solutions” are used in the sense “as many solutions as possible relative to the HBM approximation”, or in other words ”all solutions of the HBM equations”.

The harmonic balance method (HBM) is widely used in finding approximation to periodic solutions of nonlinear differential equations; the main HBM step consists of transforming the set of nonlinear differential equations into a set of nonlinear algebraic equations, which in turn can be solved to yield the Fourier coefficients of a particular solution. The HBM is a very efficient method and is capable of handling nearly all types of nonlinearities (polynomial [3], friction [4], contact [5]) regardless of their amplitudes (strong or weak nonlinearities). The Newton–Raphson algorithm and continuation methods are typically applied to the algebraic equations in order to track the solution as frequency varies [6] (note: by

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frequency we meant excitation frequency (in the forced case), and natural frequency (in the unforced case)). With a change in frequency, bifurcations may occur and several new branches of solutions can be computed from these bifurcation points depending on the specific bifurcation type [6,3]. The locations of bifurcation points and tangent directions can be computed using Newton–Raphson algorithms in a unique algebraic system (e.g. see [6]). Determining all bifurcation points however is not a simple task as some points may be easily overlooked (e.g. when the continuation step length is too high or when the determinant of the Jacobian matrix tends to zero but does not change sign). Moreover, if some branches of solutions are disconnected (in the sense that they do not arise from bifurcation points of previous branches), then the continuation algorithms would fail to detect them and new tools would be needed to solve the algebraic system of equations induced by the HBM.

Several methods for finding multiple solutions of dynamic or algebraic systems have been proposed in the literature [3,6–16]; their reviews and presentations will be provided in Section 3. We will specifically focus on the use of Groebner bases for solving the polynomial system of equations generated from the HBM. A Groebner basis can be viewed as a rewriting of the original polynomial system to be solved, yet with additional properties (e.g. in some cases in triangular form) that facilitate the resolution step. The use of a Groebner basis in the field of structural mechanics has already been proposed in a number of studies, including [17,18]. In [17], Groebner bases are used to determine multiple static equilibria of geometrically nonlinear plates subjected to static forces. Before finding solutions, the structural system is reduced using a few modal shapes and the Ritz method, thus leading to a polynomial system with a few degrees of freedom (5 max.) describing the static solution of the structural system. The author then computes a (triangular) Groebner basis of this reduced equation and solves the resulting (triangular) system by means of lifting. In [18], Groebner bases are used to compute the solution of nonlinear free vibration of geometrically nonlinear composite plates. The author also reduces the system using the Ritz method and moreover assumes a harmonic response of the structure, thereby transforming the nonlinear differential equations into a set of polynomial algebraic equations (equivalent to an HBM with 1 harmonic). Once again, Groebner bases are computed in triangular form and the system is solved by deriving an expression of the nonlinear frequency vs. motion amplitude. In both studies, the author notes that their reduced model has been limited in size due to the significant increase in Groebner basis computation time with the number of variables (e.g. see [17]: "…, it creates the need for 11 unknown constants and this cannot be accomplished with the computer available for the current studies.").

In this paper, we propose using Groebner bases [11] to solve the set of algebraic equations given by the HBM in order to find multiple steady-state solutions. Unlike previous studies, this one will allow us to identify dynamic solutions of the structural system in the forced case. In order to reduce the number of variables (and therefore the computation time) in the search for multiple solutions, we will first search for solutions of the undamped problem, and then use a damping continuation procedure to recover solutions of the damped system.

The paper will be organized as follows: Section 2 will describe the application of the harmonic balance method along with the continuation methods. Section 3 will then focus on solving the polynomial system and display the resolution method used in this paper. Lastly, Section 4 will present a numerical application on a simple example corresponding to an 8-dof cyclic structure with cubic nonlinearities. The set of all steady-state solutions will be computed by the proposed method in conjunction with a stability analysis. Solutions will be compared to the results of temporal integrations, in showing excellent agreement.

2. Harmonic balance method applied to a system with polynomial nonlinearities

In this section, we will present an application of the harmonic balance method [5,19,3] for solving a nonlinear dynamic system along with the arc length continuation algorithms [6,5] introduced to follow the solution as frequency varies. In addition, the continuation procedure on the damping parameter for transforming an undamped solution into a damped solution will be presented.

2.1. Harmonic balance method

Let us consider an n dof nonlinear dynamic system given by the following equation:

$$\mathbf{M}\ddot{\boldsymbol{u}}+\mathbf{C}\dot{\boldsymbol{u}}+\mathbf{K}\boldsymbol{u}+\boldsymbol{F}_{nl}(\boldsymbol{u})=\boldsymbol{F}(t) \tag{1}$$

where $\boldsymbol{u}(t)$ is a vector of unknown size n, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the mass, damping and stiffness matrices, $\boldsymbol{F}(t)$ is the excitation force vector which is assumed to be periodic with period $T=2\pi/\omega$, and finally $\boldsymbol{F}_{nl}(\boldsymbol{u})$ is the vector of nonlinear forces assumed to be conservative and polynomial, i.e.,

$$[\boldsymbol{F}_{nl}]_i(\boldsymbol{u})=\sum_{(\boldsymbol{\alpha})\in\mathcal{S}_i} c(\boldsymbol{\alpha})\boldsymbol{u}^{\boldsymbol{\alpha}} \tag{2}$$

where $\mathcal{S}_i\subset\mathbb{N}^n$ is the support of polynomial $[\boldsymbol{F}_{nl}]_i$. Such dynamic systems arise, for example, after the finite element modeling of mechanical systems with geometric nonlinearity [3]. This paper focuses on the (possibly multiple) steady-state solutions of equation (1) under harmonic excitation. In order to compute this solution, the harmonic balance method (HBM) [19,5] is applied; this method consists of searching for the solution in the form of a truncated Fourier series up to the H harmonic, as