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Stochastic stability of systems with semi-Markovian switching☆

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ABSTRACT

This paper examines stochastic stability of switched dynamics in continuous time. The time evolution of the so called continuous state is at all times, determined by the dynamics indexed by the switching process or discrete state. The main contribution of this paper appears as stochastic stability results for switched dynamics with semi-Markovian switching. The notion of moment stability in the wide sense (MSWS) is applied as a generalization of ϵ-moment stability. A sufficient criterion for MSWS is presented for the above class of systems, where each subsystem is assumed to be characterized by a Lyapunov function candidate together with an associated growth rate equation. For the set of Lyapunov functions, a compatibility criterion is assumed to be fulfilled, bounding the ratio between pairs of Lyapunov functions.

1. Introduction

Randomly Switched Systems (RSS) denote a class of systems in which system state evolves in a continuous state space in continuous or discrete time according to one system among a finite set of dynamics. RSS have been suggested as a suitable modeling paradigm in diverse areas such as finance, population dynamics, manufacturing, and fault-tolerant control (Cassandras & Lygeros, 2007). Stability of Markov Jump Linear Systems (MJLS), is studied in Bolzern, Colaneri, and De Nicolao (2006), Feng, Loparo, Ji, and Chizeck (1992) and Tanelli, Bolzern, and Colaneri (2010), as a special case of RSS in which dynamics are given by ordinary linear differential equations, and the switching process is a continuous time Markov chain with a discrete state space. Stability in the more general case of Switched Diffusion Processes (SDP), where process noise is represented through a Wiener process, is studied in Yuan and Mao (2003). In Leth, Schioler, Gholami, and Cocquempot (2013) the concept of moment stability in the wide sense (MSWS), which avoids any reference to stochastic convergence properties, is applied to SDP. We adopt in this paper the MSWS definition of Leth et al. (2013). MSWS relates directly to ϵ-moment stability and specifically to mean square stability for $\epsilon = 2$. More indirectly the MSWS definition as well as the Noise to State Stability of Mateos and Cortes (2013) may be seen as the target property for the Noise to State Lyapunov function (ns-lf) of Deng, Krstic, and Williams (2001). In Leth et al. (2013) continuous time Markovian switching is assumed. In this paper switching is allowed to be semi-Markovian, however no process noise is present. The extension to semi-Markov processes significantly extends the validity of results within the aforementioned application areas, e.g. population dynamics, error prone systems, etc. In Shaikhet (2011) appears the definition (9.1) *uniformly mean square bounded*. We may say that MSWS is a generalization of that to a broader class than quadratic statistics. The results obtained here are distinguished from Tanelli et al. (2010) by allowing nonlinearities and generalizing to semi-Markovian switching. Semi-Markovian switching is studied in Mariton (1989), where sojourn times are approximated by higher order Markovian models, whereas mean square stability of semi-Markovian Jump linear systems is studied in Huang and Shi (2011) and Schwartz and Haddad (2003) through rate bounding. In this paper we study generally the ϵ-moment stability of semi-Markovian Jump (non)Linear Systems without any reference to rate bounds or monotony properties as in Schwartz and Haddad (2003). The section following this introduction provides the mathematical prerequisites and principal definitions. This is followed by the analytical results section, in which the main result is presented. Finally, conclusions and discussions are provided along with suggestions for future research.

2. System definition

Throughout, we let (x, σ) denote a continuous-time stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with values in the state

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space $\mathbb{R}^n \times \mathcal{D}$ where $\mathcal{D} = \{1, \ldots, M\}$, i.e. $(x, \sigma) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathcal{D}$. To simplify notation, we generally write $x(t)$ and $\sigma(t)$ in place of $x(\cdot, t)$ and $\sigma(\cdot, t)$, respectively. The pair (x, σ) is referred to as the state, x as the continuous state, and σ as the discrete state or switching process. Moreover, we assume throughout that σ is a right continuous semi-Markov process. For continuous time, the evaluation of x is based on system dynamics in the shape of an Ordinary Differential Equation (ODE). More precisely, for each discrete state, $d \in \mathcal{D}$ an ODE is defined

$$\frac{d}{dt}x(t) = f_d(x(t)) \tag{1}$$

where $f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n$ is an appropriate mapping satisfying suitable regularity conditions to ensure unique continuous solutions. The overall switched dynamics may therefore be written as

$$\frac{d}{dt}x(t) = f_{\sigma(t)}(x(t)). \tag{2}$$

We say that a functional $V : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function (candidate) if it is twice continuously differentiable, and $V^{-1}([0, C])$ are compact sets for all $C \geq 0$. For each $d \in \mathcal{D}$, it is assumed that there exist a real number λ_d and a Lyapunov function candidate $V_d : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\nabla V_d(x) f_d(x) \leq \lambda_d V_d(x), \tag{3}$$

with ∇V_d denoting the gradient of V_d. For $\lambda_d < 0$, inequality (3) ensures stability of (1). We generally do not assume $\lambda_d < 0$, since we allow instability for a subset of discrete states. We shall assume, as in Yang, Jiang, and Cocquempot (2009), that the Lyapunov function candidates V_d are compatible, i.e. a real number $\mu > 1$ exists such that

$$V_d(x) \leq \mu V_b(x), \quad \forall d, b \in \mathcal{D}, \ \forall x \in \mathbb{R}^n. \tag{4}$$

The switching process σ governs the choice of smooth dynamics for the continuous state and is assumed to be a right continuous semi-Markov process with transition instants $\{t_j\}$. Let $\tau_n = t_n - t_{n-1}$ and $\sigma_n = \sigma(t_n)$ then, by the semi-Markov property, the discrete time process $\{\sigma_n, \tau_n\} = \{\sigma(t_n), \tau_n\}$ is a so called Markovian renewal process, i.e. let $\{\mathcal{Q}_n\}$ be its natural filtration then

$$P((\sigma_{n+k}, \tau_{n+k}) \in A | \mathcal{Q}_n) = P((\sigma_{n+k}, \tau_{n+k}) \in A | \sigma_n). \tag{5}$$

Under time invariance we may therefore characterize the process completely by its one step conditional probability $P((\sigma_{n+1}, \tau_{n+1}) \in A | \sigma_n)$. Next we define the conditional sojourn time distribution $m_{s,s'}$ given present and future state through

$$\begin{aligned} m_{s,s'}(A) &= P(\tau_{n+1} \in A | \sigma_{n+1} = s', \sigma_n = s) \\ &= P(\tau_{n+1} \in A, \sigma_{n+1} = s' | \sigma_n = s)/P_{s,s'}, \end{aligned} \tag{6}$$

for all $s, s' \in \mathcal{D}$ and with $P_{s,s'} = P(\sigma_{n+1} = s' | \sigma_n = s)$.

3. Stochastic stability

Stability properties of $x(t)$ need to be established in the context of stochastic stability, in which a variety of inter-related definitions exist. Most definitions are based on associated definitions of convergence such as convergence in probability, convergence in mean/moment and almost sure convergence.

We define the system (2) to be *moment-stable in the wide sense* (MSWS) if there exist $0 \leq K < \infty$, $0 < \epsilon < 1$ and Lyapunov function candidates $\{V_d \mid d \in \mathcal{D}\}$ such that:

$$E[V^{\epsilon}_{\sigma(t)}(x(t))] \leq K, \quad \forall t \geq 0 \tag{7}$$

whenever $E[V^{\epsilon}_{\sigma(0)}(x(0))] < \infty$.

4. Stability analysis

Stability analysis is based on the definition of a dominating process, U, for which stability criteria are given. Since U is an approximation from above, the presented criteria can at most be sufficient.

Now in Leth et al. (2013) it is shown that

$$V_{\sigma(t)}(x(t)) \leq U(t), \quad \forall t \geq 0 \tag{8}$$

where, for a given $0 < \epsilon < 1$, the process U is defined by

$$\begin{aligned} &\frac{d}{dt}U(t) = \epsilon \lambda_{d_i} U(t), \quad t \in [t_i, t_{i+1}), \\ &U(t_{i+1}) = \mu^{\epsilon} U(t^-_{i+1}), \\ &U(0) = V^{\epsilon}_{\bar{\sigma}(0)}(\bar{x}) \end{aligned}$$

with $\{t_j\}$ the sequence of transition instants of a particular realization $\bar{\sigma}$ of the switching process σ with $\bar{\sigma}(t) = d_i$ for $t \in [t_i, t_{i+1})$.

By (8) we therefore obtain

$$E[U(t)] \geq E[V^{\epsilon}_{\sigma(t)}(x(t))], \tag{9}$$

so that bounding $E[U(t)]$ globally leads to MSWS. Now define $U_n = U(t_n)$, then by definition

$$U_{n+1} = \mu^{\epsilon} U_n \exp(\epsilon \lambda_{d_n} \tau_{n+1}).$$

By construction, $\{U_n\}$ is adapted to $\{\mathcal{Q}_n\}$, so from (5) the following independence relation holds true:

$$\begin{aligned} &P(\tau_{n+1} \in A, \sigma_{n+1} = s' | \sigma_n = s, U_n \in B) \\ &\quad = P(\tau_{n+1} \in A, \sigma_{n+1} = s' | \sigma_n = s) = m_{s,s'}(A) P_{s,s'}. \end{aligned} \tag{10}$$

Let $\mathfrak{m}_{n,s}$ denote the state conditional distribution of U_n, then (10) allows us to write

$$\begin{aligned} &P(\tau_{n+1} \in A, \sigma_{n+1} = s', \sigma_n = s, U_n \in B) \\ &\quad = m_{s,s'}(A) P_{s,s'} \mathfrak{m}_{n,s}(B) P(\sigma_n = s). \end{aligned} \tag{11}$$

For each $d \in \mathcal{D}$ define the processes $\gamma_d(n) = I_{\sigma_n = d} U(n)$. Then

$$\gamma_{s'}(n+1) = I_{\sigma_{n+1} = s'} \sum_{s \in \mathcal{D}} I_{\sigma_n = s}\, \mu^{\epsilon} U(n)\ \exp(\epsilon \lambda_s \tau_{n+1}).$$

Taking expectation according to (11), and writing $P_s = P(\sigma_n = s)$

$$\begin{aligned} E[\gamma_{s'}(n+1)] &= \sum_{s \in \mathcal{D}} \int \mu^{\epsilon} \exp(\epsilon \lambda_s \tau)\, dm_{s,s'}(\tau) P_{s,s'} \int u\, d\mathfrak{m}_{n,s}(u) P_s \\ &= \mu^{\epsilon} \sum_{s \in \mathcal{D}} \mathcal{L}_{s,s'}(\epsilon \lambda_s) P_{s,s'}\, E[\gamma_s(n)] \end{aligned} \tag{12}$$

where $\mathcal{L}_{s,s'}$ is the Laplace transform of the state dependent distribution $m_{s,s'}$ (moment generating function). For further development $\mathcal{L}_{s,s'}(\epsilon \lambda_s)$ needs to be differentiable at $\epsilon = 0$, which, e.g. is fulfilled for exponential sojourn times and positive λ_s as well as for finite expectation and negative λ_s.

The expectation dynamics (12) may be written more compactly as

$$E[\gamma(n+1)] = \Lambda(\epsilon) E[\gamma(n)], \tag{13}$$

where $\gamma(n)$ is the M-dimensional vector with entries $\gamma_s(n)$, and $\Lambda = \Lambda(\epsilon) = \mu^{\epsilon} \left[\mathcal{L}_{s,s'}(\epsilon \lambda_s) P_{s,s'}\right]$ is the $M \times M$ matrix with row–column index (s', s).

4.1. Moment stability in the wide sense

Since by construction $E[U_n] = \sum_{d \in \mathcal{D}} E[\gamma_d(n)]$, boundedness of $\gamma(n)$ through (9) leads to a bounded $E[V^{\epsilon}_{\sigma(t)}(x(t))]$. Conditions under which $\gamma(n)$ has bounded mean will be given as a result of Lemma 1.